

# Methods of Partial Wave Analysis: Mesons and Baryons

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The spectrum of well separated and relatively narrow hadron states is well known. New states can be discovered from weak signals extracted from an analysis of **a high statistical data** and/or **reactions with many body** final states. These states are expected to be broad and **overlap with other resonances and nonresonant contributions**.

**Therefore a modern Partial Wave Analysis should:**

1. be fast enough to deal with modern high statistic data.
2. be able to perform an analysis of many data sets simultaneously.
3. be able to deal with many body final states.
4. To reveal overlapping broad states, it should respect 2 body and if possible three body unitarity.
5. To define position and properties of states, it should respect analyticity of the amplitude in the whole complex plane.

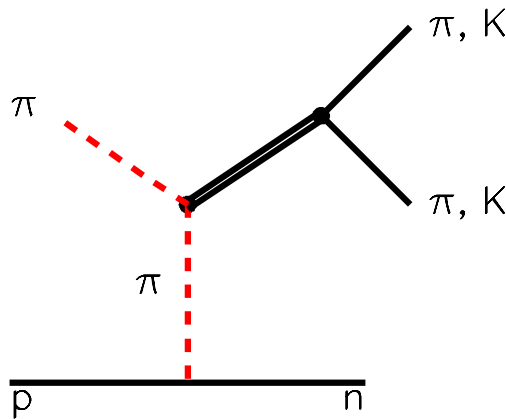
## Energy independent partial wave analysis

In this approach the angular dependence is analyzed independently on the energy dependence of the scattering amplitude.

Scattering of two particles with spin 0:

$$A(s, t) = 16\pi \sum_{\ell=0}^{\infty} (2\ell + 1) P_{\ell}(z) A_{\ell}(s) \quad s = 4m^2 - \frac{2t}{1-z}$$

However in reality we do not have such ideal case. It is necessary to analyze a more complicated process.



## CERN-Munich approach

The CERN-Munich model was developed for the analysis of the data on  $\pi^- p \rightarrow \pi^+ \pi^- n$  reaction and based partly on the absorption model but mostly on the phenomenological observations.

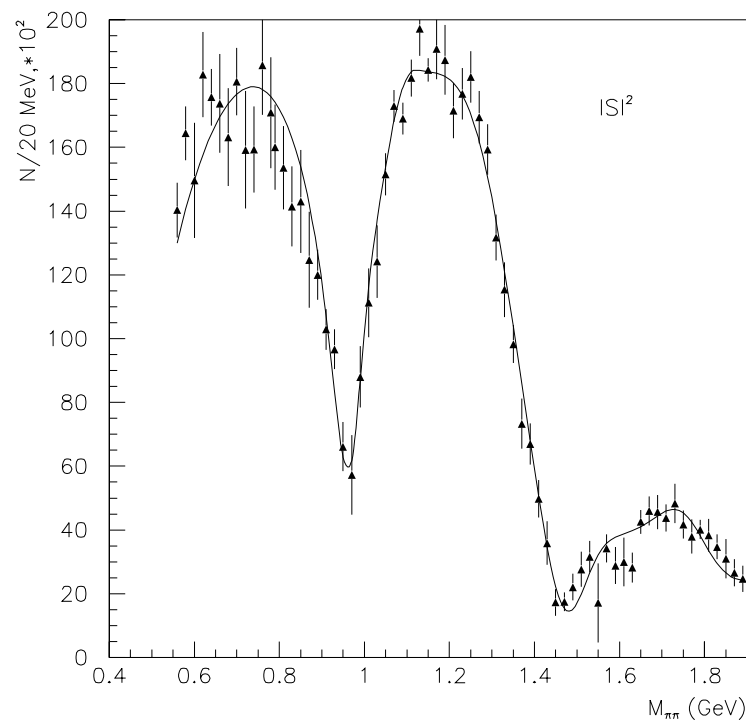
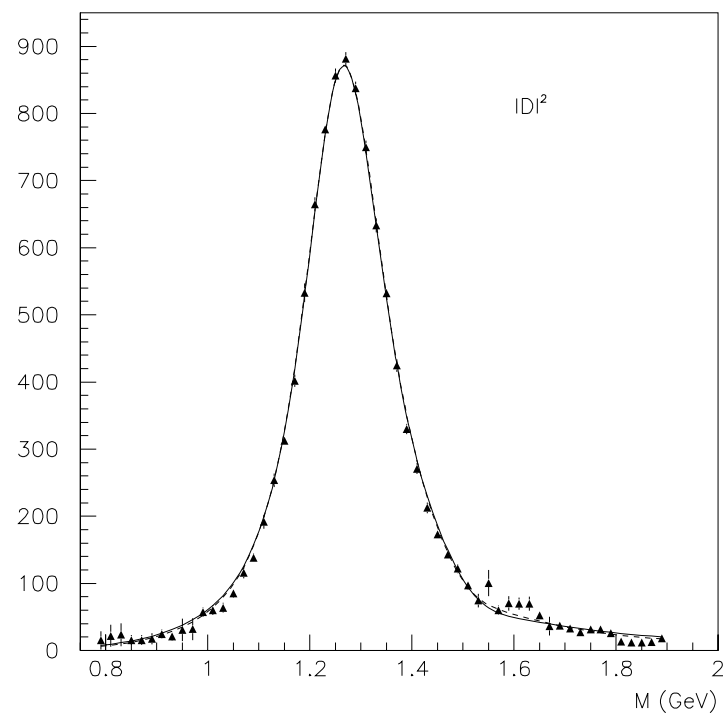
$$|A^2| = \left| \sum_{J=0} A_J^0 Y_J^0 + \sum_{J=1} A_J^- \sqrt{2} \operatorname{Re} Y_J^1 \right|^2 + \left| \sum_{J=1} A_J^+ \sqrt{2} \operatorname{Im} Y_J^1 \right|^2$$

**Additional assumptions:**

$$A_J^{(-)} = A_J^{(+)} \quad A_J^{(-)} = \frac{A_J^0}{C_J \sum_{n=0}^3 b_n M^n},$$

**GAMS, VES, BNL** approach: same expression but no assumptions about relations between amplitudes.

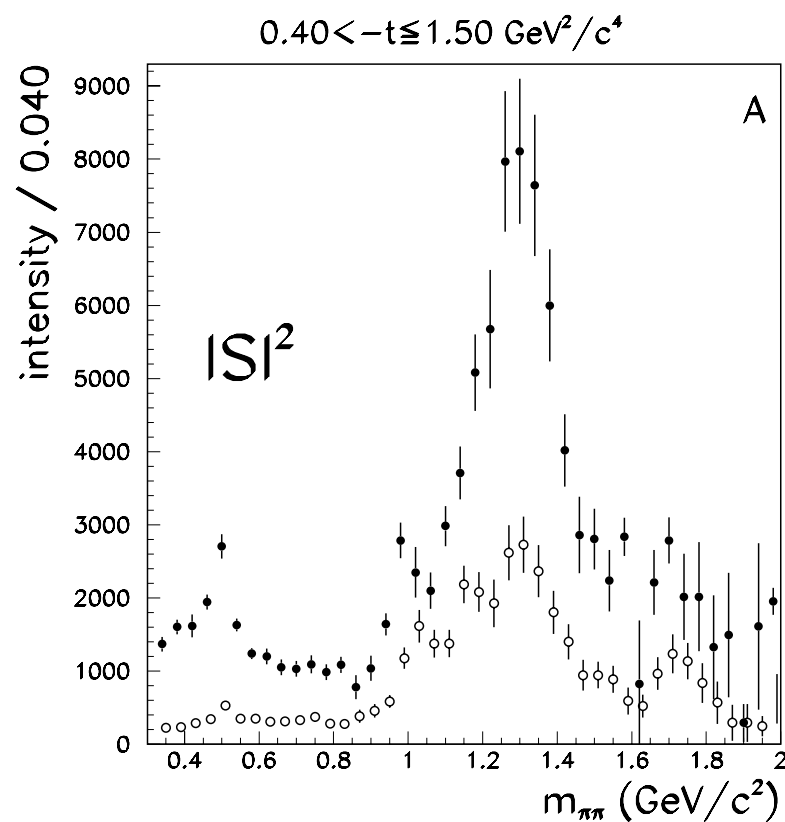
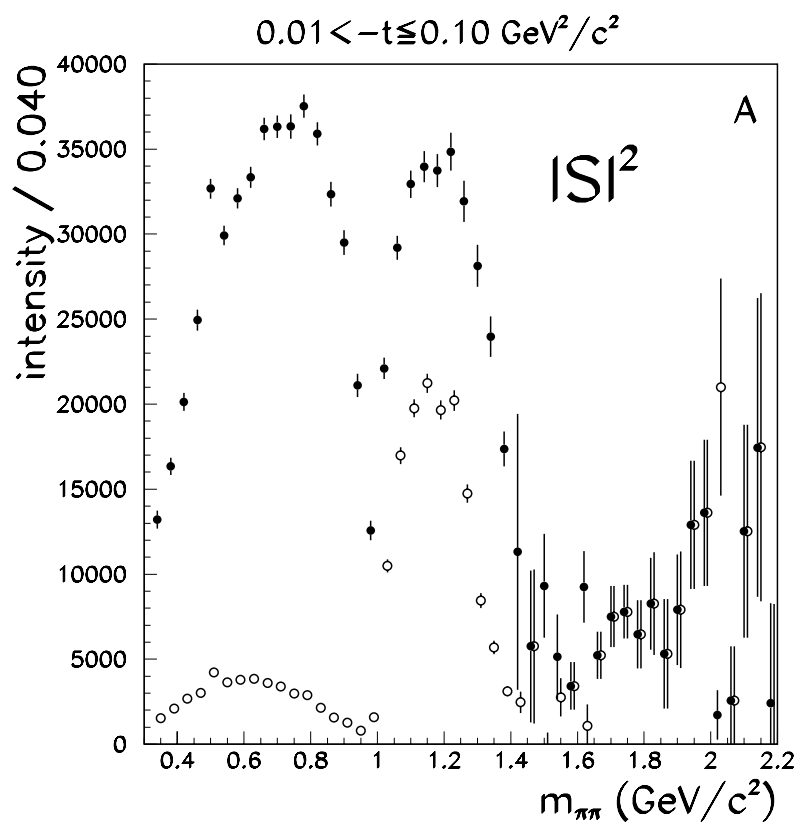
The  $\pi\pi \rightarrow \pi\pi$  D and S-wave intensities extracted by the GAMS collaboration from the reaction  $\pi^- p \rightarrow \pi^0 \pi^0 n$  at large momentum of initial pion (38 GeV/c).



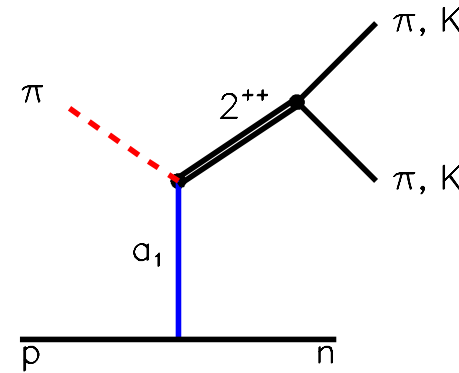
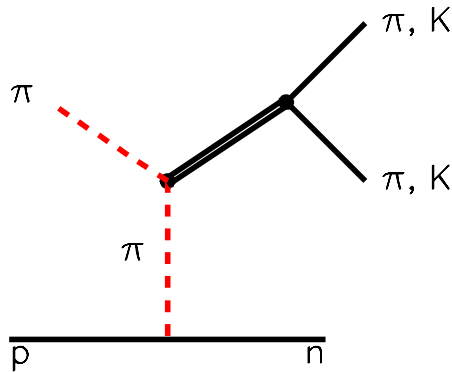
The data are extracted under assumption about dominance of  $\pi$ -exchange at small energy transferred and were confirmed only after experiments with polarized protons.

## BNL analysis

The S-wave has a very prominent structure **at large  $|t|$** .



## Reggeized exchanges:



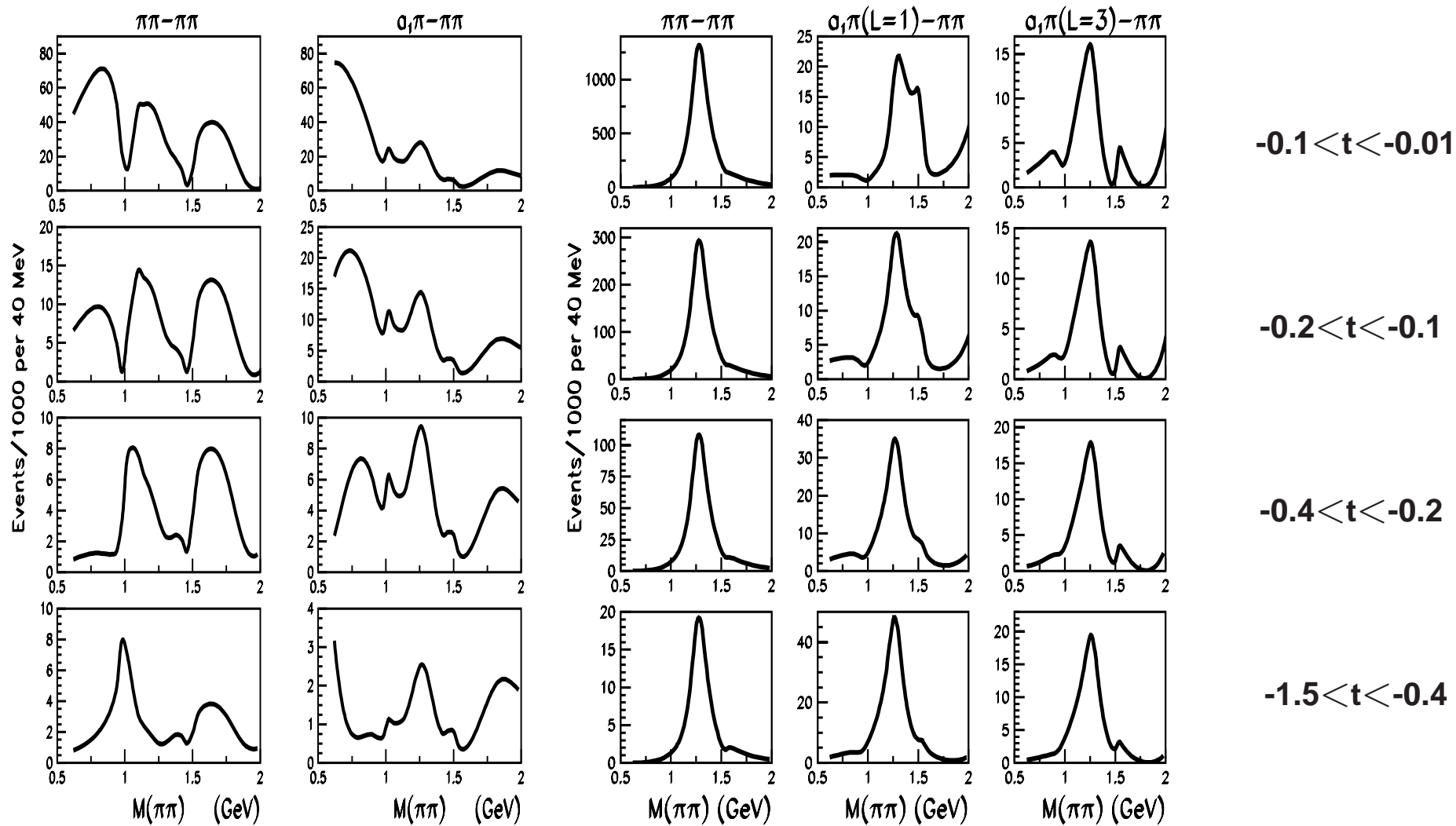
$$A_{\pi p \rightarrow \pi \pi n}^{(\text{pion trajectories})} = \sum_{\pi_j} A(\pi \pi_j \rightarrow \pi \pi) R_{\pi_j}(s_{\pi N}, q^2) (\varphi_n^+ (\vec{\sigma} \vec{p}_\perp) \varphi_p) g_{pn}^{(\pi_j)}.$$

$$A_{\pi p \rightarrow \pi \pi n}^{(a_1 \text{-trajectories})} = \sum_{a_1^{(j)}} A(\pi a_1^{(j)} \rightarrow \pi \pi) R_{a_1^{(j)}}(s_{\pi N}, q^2) (\varphi_n^+ (\vec{\sigma} \vec{n}_z) \varphi_p) g_{pn}^{(a_1^{(j)})}.$$

$$R_{\pi_j}(s_{\pi N}, q^2) = \exp\left(-i \frac{\pi}{2} \alpha_{\pi}^{(j)}(q^2)\right) \frac{(s_{\pi N}/s_{\pi N 0})^{\alpha_{\pi}^{(j)}(q^2)}}{\sin\left(\frac{\pi}{2} \alpha_{\pi}^{(j)}(q^2)\right) \Gamma\left(\frac{1}{2} \alpha_{\pi}^{(j)}(q^2) + 1\right)}$$

$$R_{a_1^{(j)}}(s_{\pi N}, q^2) = i \exp\left(-i \frac{\pi}{2} \alpha_{a_1}^{(j)}(q^2)\right) \frac{(s_{\pi N}/s_{\pi N 0})^{\alpha_{a_1}^{(j)}(q^2)}}{\cos\left(\frac{\pi}{2} \alpha_{a_1}^{(j)}(q^2)\right) \Gamma\left(\frac{1}{2} \alpha_{a_1}^{(j)}(q^2) + \frac{1}{2}\right)}$$

## S and D-waves at different t-intervals. Solution 2



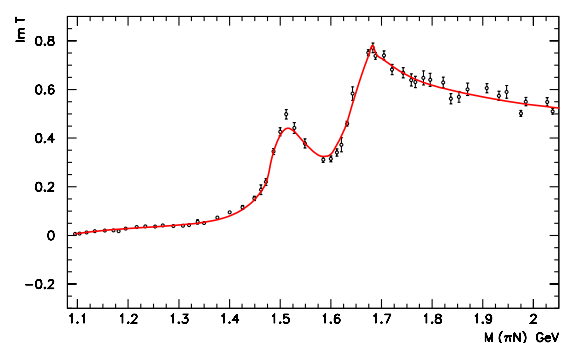
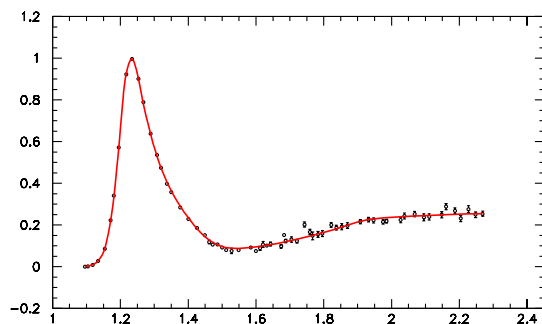
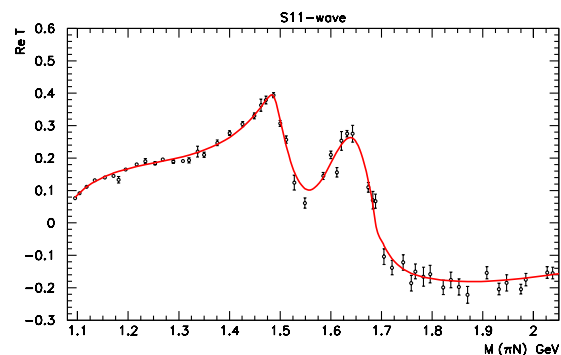
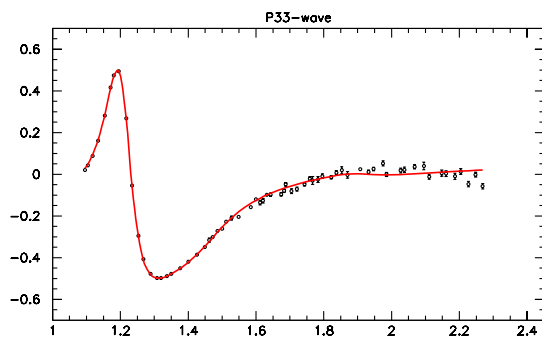


The  $\pi N$  elastic scattering is a perfect two particle reaction.

But... nucleon spin should be measured:

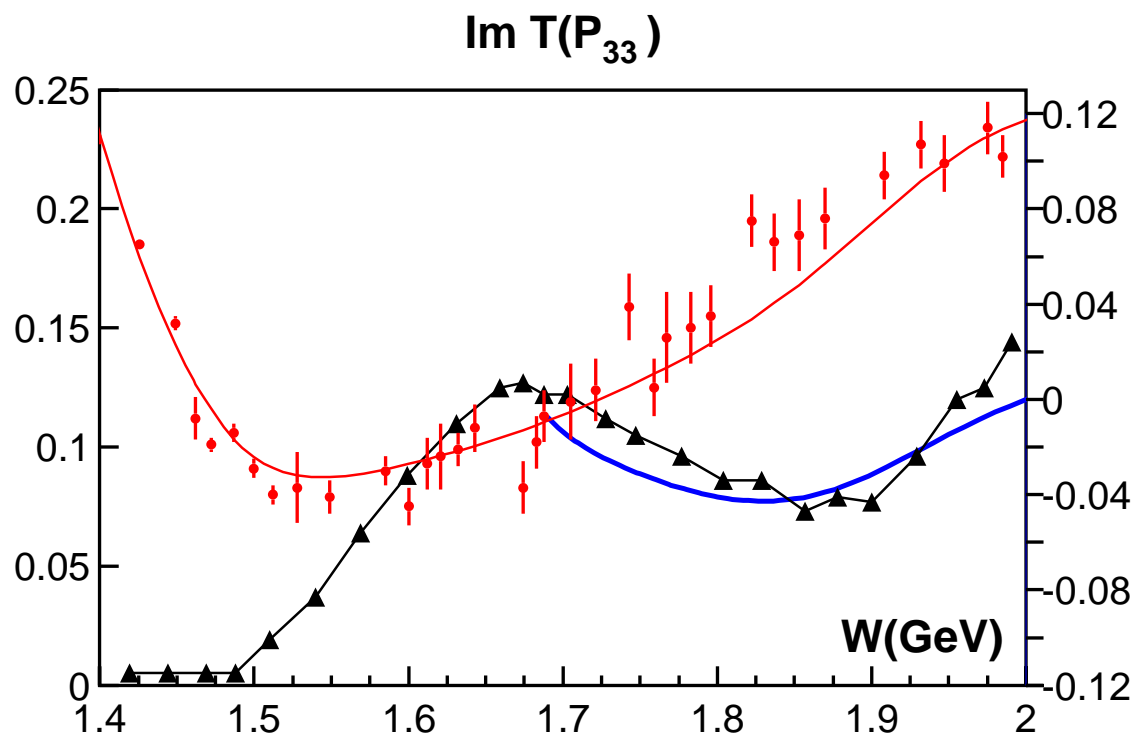
$$G(s, t) = \sum_L \left[ (L+1)F_L^+(s) - LF_L^-(s) \right] P_L(z)$$

$$A_{\pi N} = \omega^* \left[ G(s, t) + H(s, t)i(\vec{\sigma}\vec{n}) \right] \omega', \quad H(s, t) = \sum_L \left[ F_L^+(s) + F_L^-(s) \right] P'_L(z)$$



The strong signals can be extracted from existing data imposing dispersion relations.

But there are ambiguities in the extraction of weak signals.



which can lead to the different interpretation of hadron spectrum, e.g. absence or a proof for  $P_{33}(1600)$  state.

The analysis of energy dependence in many cases is not simpler after energy independent partial wave decomposition. Correlations between energy dependence and angular dependence is not controlled.

## Energy dependent approach

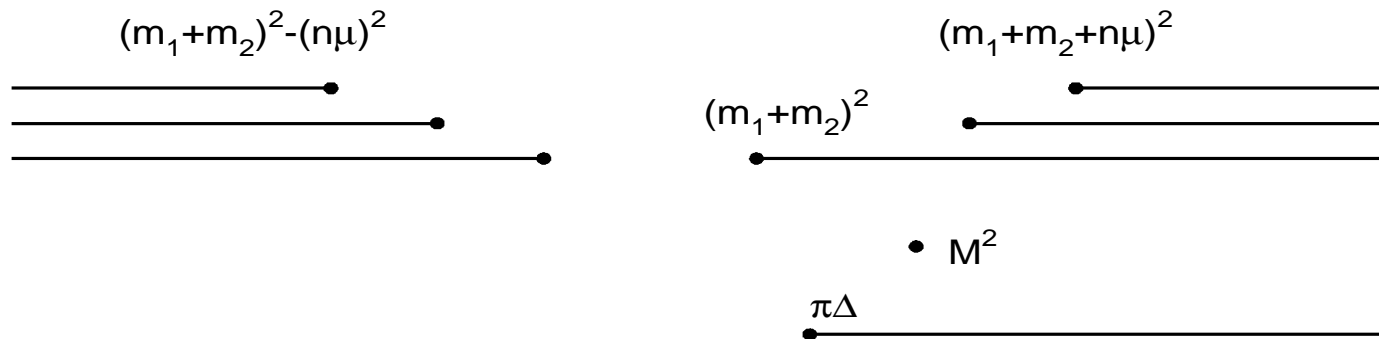
In many reactions an unambiguous decomposition at fixed energies is impossible. Then energy part and angular part are analyzed together:

$$A(s, t) = \sum_{\beta\beta'n} A_n^{\beta\beta'}(s) Q_{\mu_1 \dots \mu_n}^{(\beta)+} F_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n} Q_{\nu_1 \dots \nu_n}^{(\beta')}$$

1. C. Zemach, Phys. Rev. 140, B97 (1965); 140, B109 (1965).
2. S.U.Chung, Phys. Rev. D 57, 431 (1998).
3. A.V. Anisovich *et al.* J. Phys. G 28 15 (2002)  
 V. V. Anisovich, M. A. Matveev, V. A. Nikonov, J. Nyiri and A. V. Sarantsev,  
*Hackensack, USA: World Scientific (2008) 580 p*
  1. Correlations between angular part and energy part are under control.
  2. Unitarity and analyticity can be introduced from the beginning.
  3. However to fix simultaneously energy and angular dependencies of the amplitude a combined fit of many reactions is needed.

The Partial Wave Amplitude has the following singularities:

1. **Pole singularities**: stable particles and resonances.
2. **Threshold (square root) singularities** defined by the decay of the system into final particles.
3. **Logarithmical singularities** due to rescattering of three particles (triangle diagrams).
4. **Box singularities** (one over square root) defined by 4 particle rescattering (box diagrams).
5. cuts on left-hand side complex plane due to exchange processes.



**The simplest parameterization of the pole, Breit-Wigner amplitude:**

$$A = \frac{\Lambda}{M^2 - s - iM\Gamma}$$

**The pole is at  $s = M^2 - iM\Gamma$ . The residue in the pole  $R = \Lambda$ , the amplitude has a peak at  $s = M^2$ .**

**The width of the state is formed by decays into open channels. Then the threshold singularities should be taken into account:**

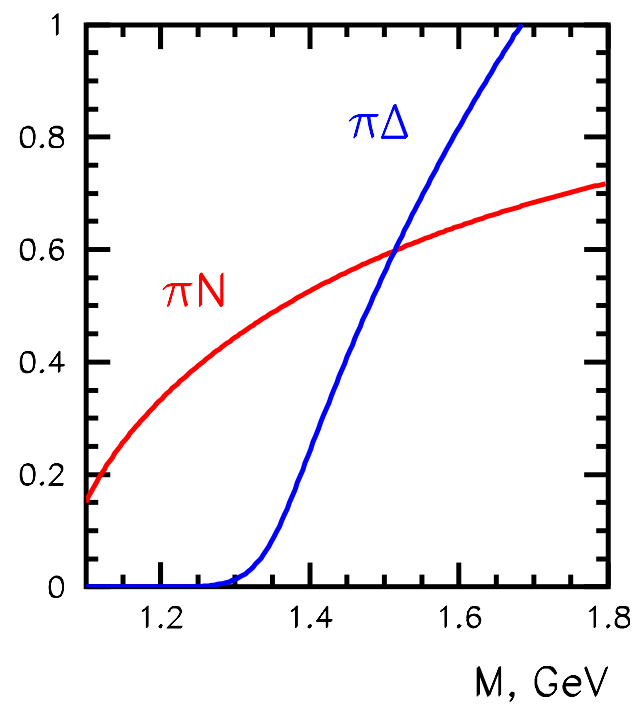
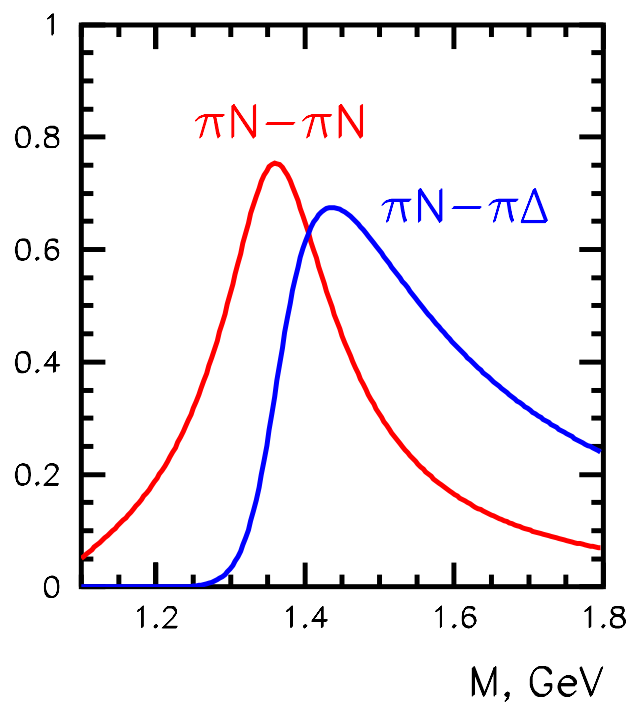
$$A_{ab} = \frac{g_a g_b}{M^2 - s - i \sum_j \rho_j(s) g_j^2}$$

**where  $\rho_j(s)$  is the phase volume.**

The peak is shifted in different reactions:

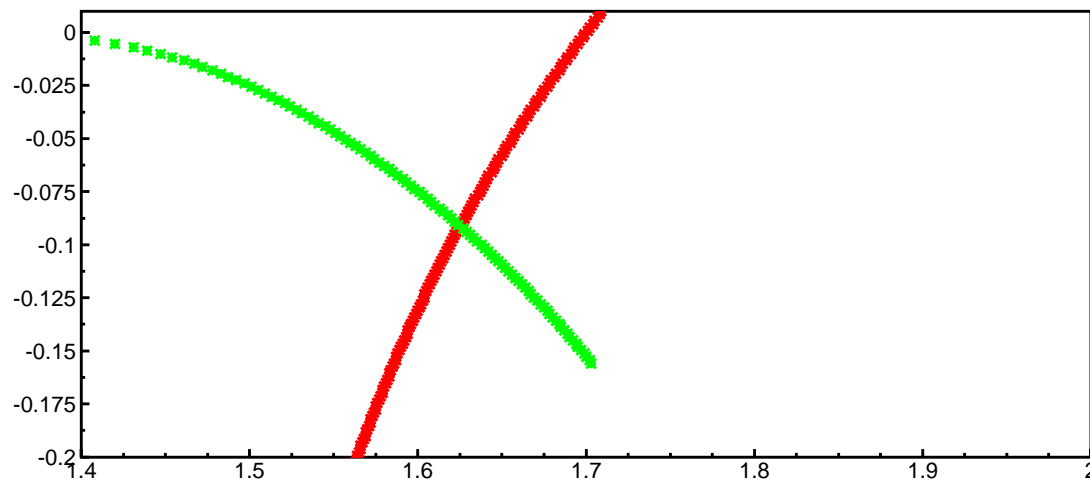
Example:  $P_{11}$  state decaying into  $\pi N$  and  $\pi\Delta(1232)$  systems.  $M_{BW} = 1.45\text{GeV}$ .

The pole position at 1300 MeV.



In the case of fast increasing phase volume the amplitude pole is notably shifted from the Breit-Wigner mass.

One pole one channel ( $\pi\Delta$ ) K-matrix.



And pole residues are complex numbers:

$$\frac{1}{2\pi i} \oint ds A_{ii}(s) = \frac{g_i^2}{1 + i \sum_{\alpha} g_{\alpha}^2 \rho'_{\alpha}(s)}$$

# 1 $K$ -matrix representation of the scattering amplitude

The unitarity condition for the partial wave amplitude is:

$$SS^+ = I \quad S = I + 2i\hat{\rho}(s)\hat{A}(s)$$

$$S = \frac{I + i\hat{\rho}\hat{K}}{I - i\hat{\rho}\hat{K}} = I + 2i\hat{\rho}A(s), \quad A(s) = \hat{K}(I - i\hat{\rho}\hat{K})^{-1}$$

Where  $\hat{K}$  is a real matrix.

**Can one obtain the K-matrix from basic principles only?**

1. The amplitude is symmetric for transition between final states, thus  $K$ -matrix must be also symmetric.
2. The amplitude must have pole singularities of the first order: K-matrix also can have only first order poles.



One pole, many channel K-matrix corresponds to the relativistic Breit-Wigner amplitude:

$$K_{ab} = \frac{g_a g_b}{M^2 - s} \quad \rightarrow \quad A_{ab} = \frac{g_a g_b}{M^2 - s - i \sum_j \rho_j(s) g_j^2}$$

For two poles:

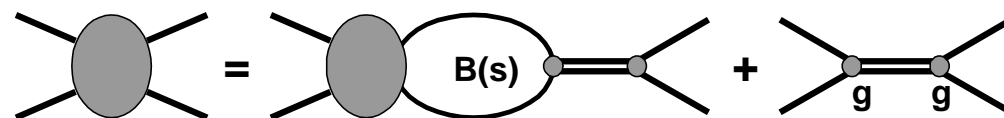
$$K = \frac{g_1^2}{M_1^2 - s} + \frac{g_2^2}{M_2^2 - s}$$

$$A(s) = \frac{g_1^2(M_2^2 - s) + g_2^2(M_1^2 - s)}{(M_1^2 - s)(M_2^2 - s) - i\rho(s)(g_1^2(M_2^2 - s) + g_2^2(M_1^2 - s))}$$

Compare with sum of two Breit-Wigner amplitudes:

$$A(s) = \frac{g_1^2(M_2^2 - s) + g_2^2(M_1^2 - s) - 2i\rho(s)g_1^2g_2^2}{(M_1^2 - s)(M_2^2 - s) - \rho^2(s)g_1^2g_2^2 - i\rho(s)(g_1^2(M_2^2 - s) + g_2^2(M_1^2 - s))}$$

The K-matrix amplitude can be considered as a solution of Bethe-Salpeter equation:



$$A_{ab}(s, s) = \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{A_{aj}(s, s') i \rho_j(s') K_{jb}(s)}{s' - s - i0} + K_{ab}(s)$$

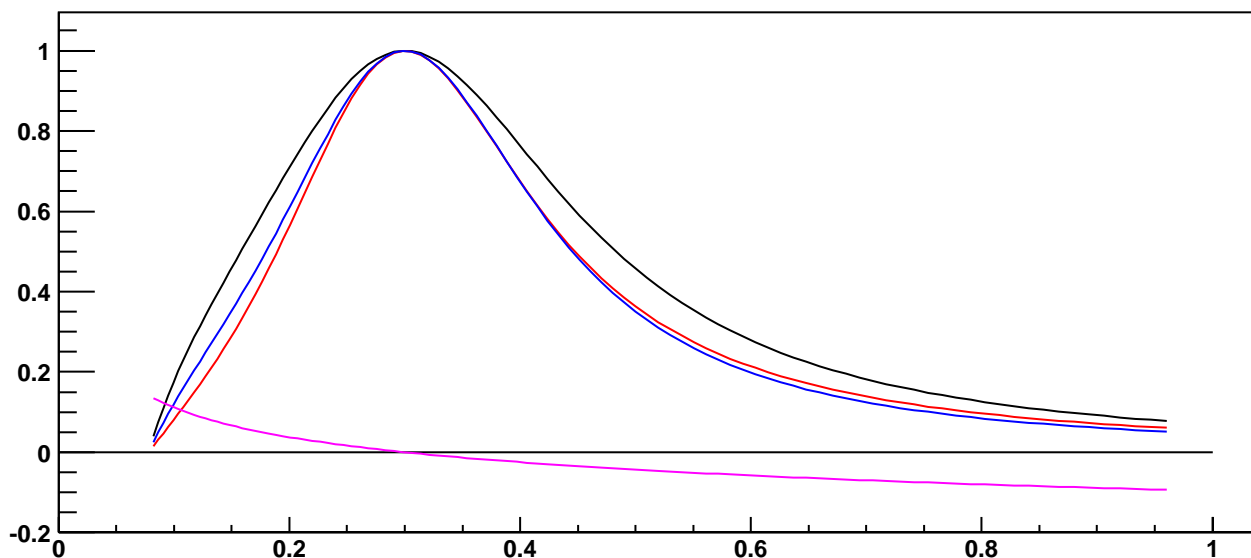
But ... with omitted real part of loop diagrams:

$$A_{ab} = A_{aj} i \rho_j(s) K_{jb} + K_{ab} \quad \rightarrow \quad \hat{\mathbf{A}} = \hat{\mathbf{K}} (\mathbf{I} - i \hat{\rho} \hat{\mathbf{K}})^{-1}$$

$$i\rho(s) = i\sqrt{\frac{s - 4m^2}{s}} \rightarrow -\infty \quad s \rightarrow 0$$

$$i\rho(s)\left(1 - \frac{i}{\pi} \ln \frac{1 - \rho(s)}{1 + \rho(s)}\right) = i\sqrt{\frac{s - 4m^2}{s}} \left(1 - \frac{2}{\pi} \operatorname{arctg} \frac{4m^2 - s}{s}\right) \rightarrow \text{const}$$

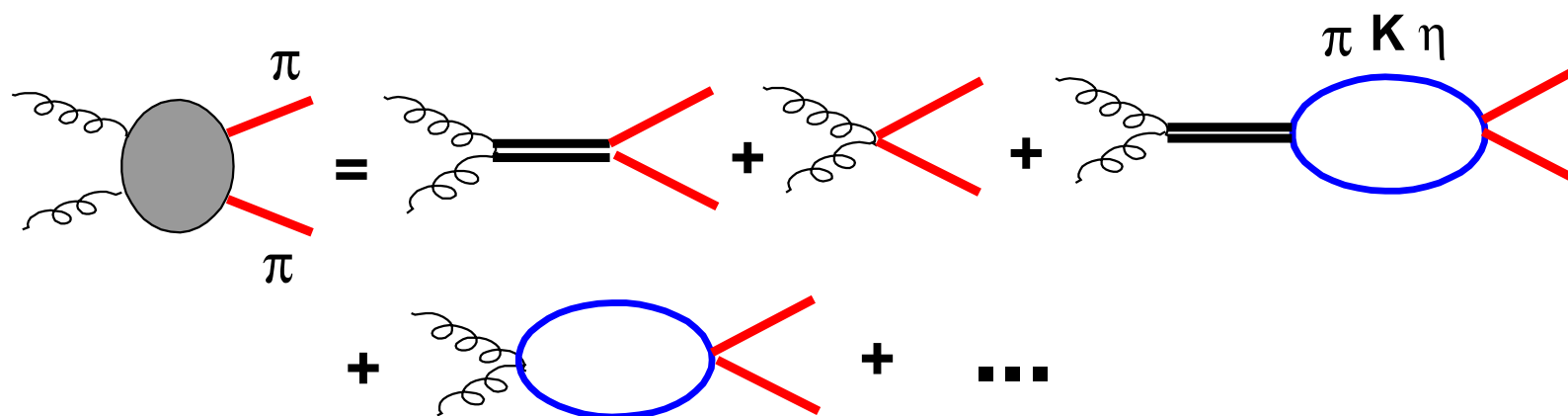
Graph



**Black curve - BW amplitude, red curve - full B(s) calculation, blue curve - BW amplitude with reduced width, magenta - dispersion correction of the real part.**

### P-vector approach

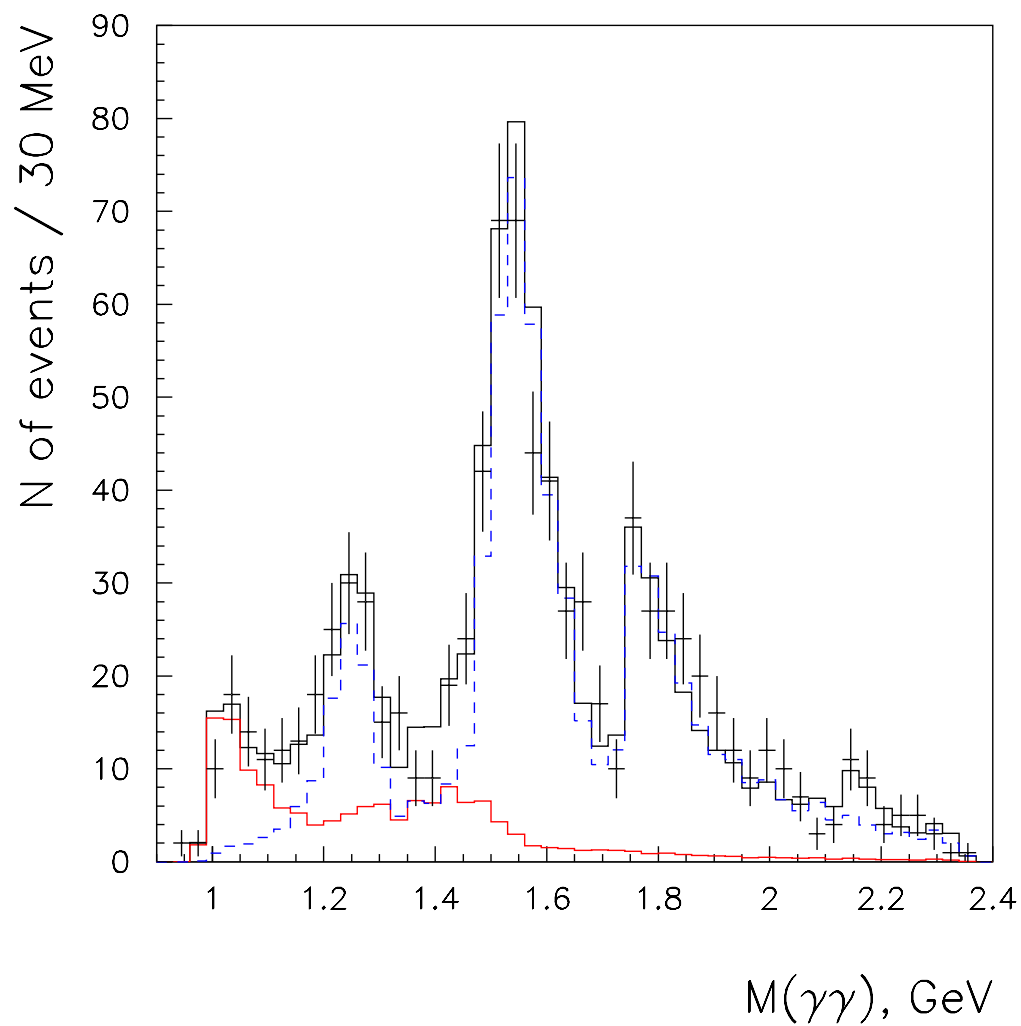
The  $\gamma\gamma \rightarrow \pi\pi$  reaction: the contribution from the  $\gamma\gamma$ -loop to the width of the state can be neglected.



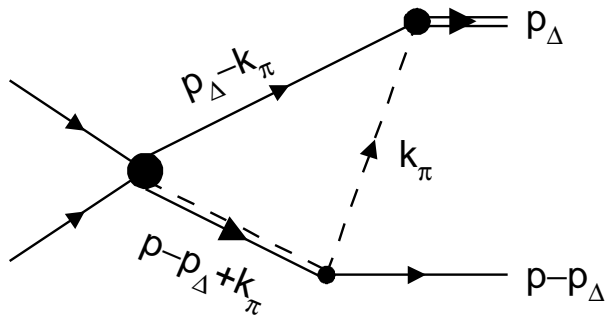
$$A_k = P_j (I - i\rho K)_{jk}^{-1} \quad P_j = \sum_m \frac{\Lambda_n g_1^{(n)}}{M_n^2 - s} + F_j$$

## Analysis of $\gamma\gamma \rightarrow K_s K_s$ (L3) reaction.

The K-matrix parameters are fixed from the fit of  $\pi\pi \rightarrow \pi\pi$  and  $\pi\pi \rightarrow KK$  data.



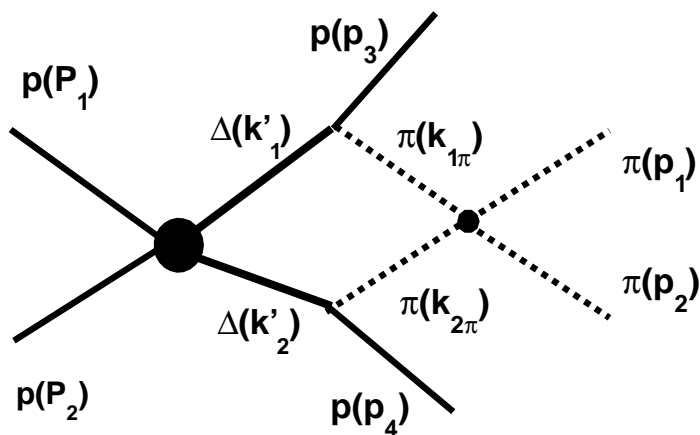
Production of three particle final states. Triangle singularities in the three particle system is **demanded by 3-particle unitarity**: this is a **logarithmic singularity**.



$$A_{\text{triangle}}^{\text{spinless}}(W^2, s) = \int \frac{d^4 k_\pi}{i(2\pi)^4} \frac{1}{m_\pi^2 - k_\pi^2 - i0} \times \frac{1}{m_\Delta^2 - (p - p_\Delta + k_\pi)^2 - im_\Delta \Gamma_\Delta} \frac{1}{m_N^2 - (p_\Delta - k_\pi)^2 - i0},$$

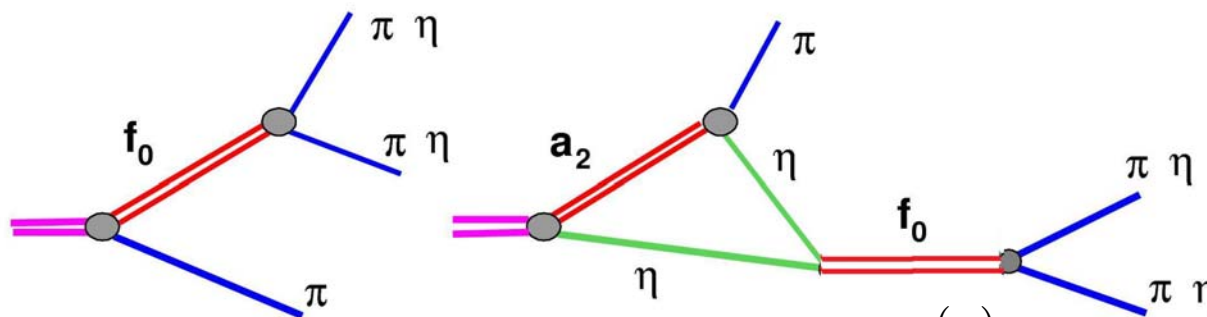
$$p = p_1 + p_2, \quad p^2 = W^2, \quad p_\Delta^2 = s, \quad W_{\text{min}} = m_N + m_N + m_\pi.$$

**Stronger singularities** (with a  $(s - s_0)^{-1/2}$  behaviour) are related to box diagrams.



Could it be that some states we observe in four particle final state reactions are box singularities???

The proton-antiproton annihilation at rest.



$$A_k = P_j (I - i\rho K)_{j1}^{-1}$$

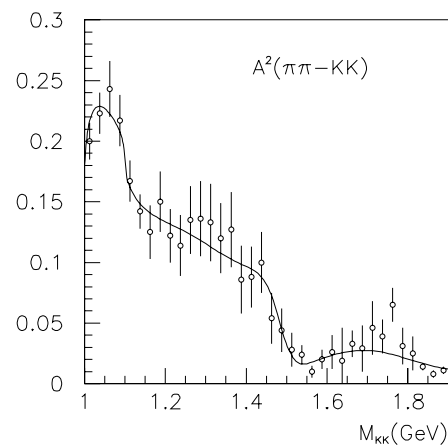
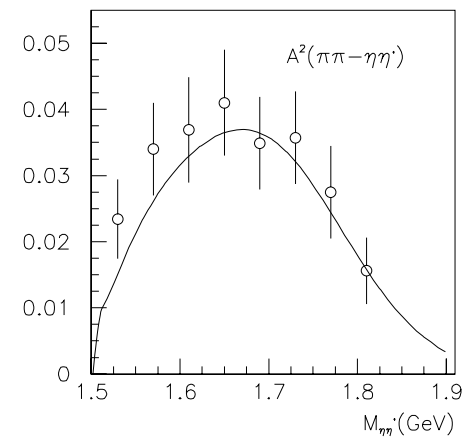
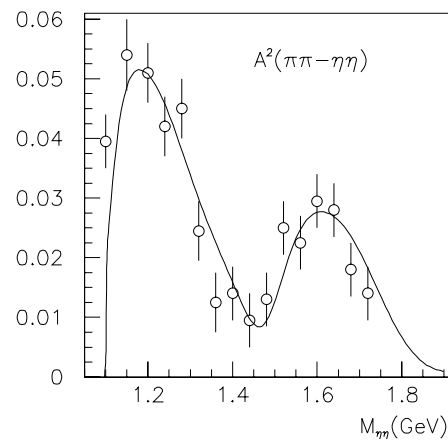
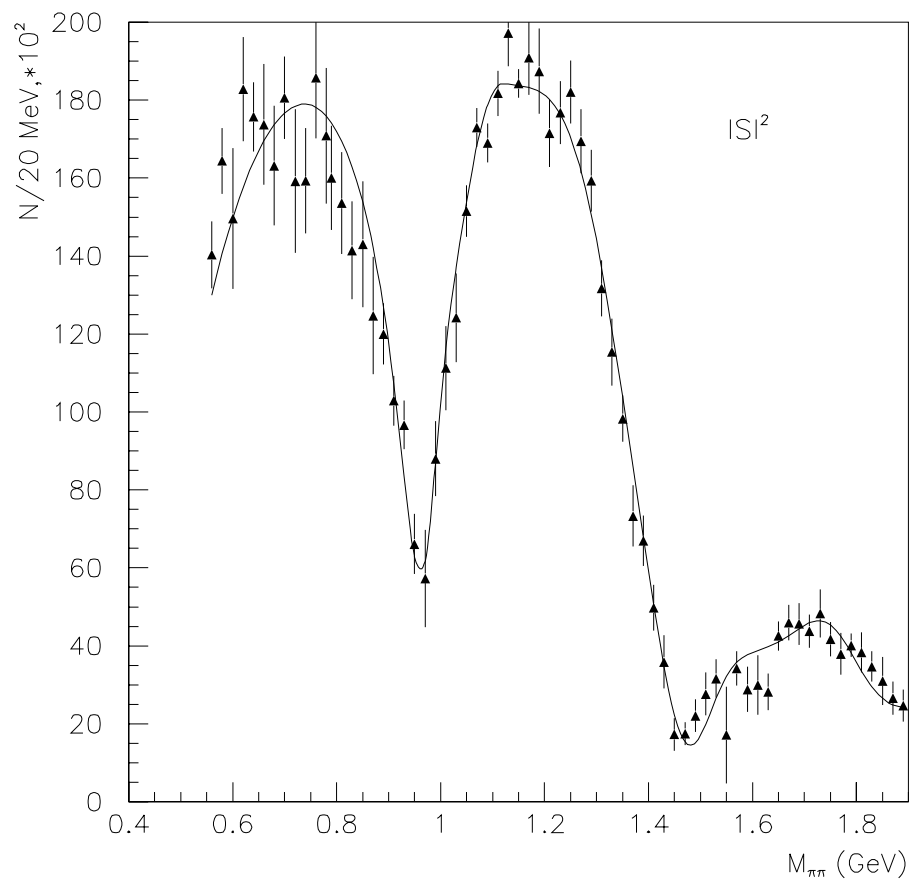
$$P_j = \sum_m \frac{\Lambda_n g_1^{(n)}}{M_n^2 - s} + F_j$$

However  $\Lambda_n$  and  $F_i$  are (at least) complex parameters.

Same P-vector would describe the large number of amplitudes. For example:

- |            |                                    |                                |
|------------|------------------------------------|--------------------------------|
| $\bar{p}p$ | $\rightarrow \pi^0 \pi^0 \pi^0$    | $\pi\pi \rightarrow \pi\pi$    |
| $\bar{p}p$ | $\rightarrow (\eta\eta)\pi^0$      | $\pi\pi \rightarrow \eta\eta$  |
| $\bar{p}p$ | $\rightarrow (\pi^+ \pi^-)\pi^0$   |                                |
| $\bar{p}p$ | $\rightarrow (K_0 \bar{K}_0)\pi^0$ | $\pi\pi \rightarrow K \bar{K}$ |
| $\bar{p}p$ | $\rightarrow (K^+ K^-)\pi^0$       |                                |

## Description of $\pi\pi \rightarrow \pi\pi$ , $\pi\pi \rightarrow \eta\eta$ , $\pi\pi \rightarrow \eta\eta'$ and $\pi\pi \rightarrow K\bar{K}$ S-wave intensity (GAMS).



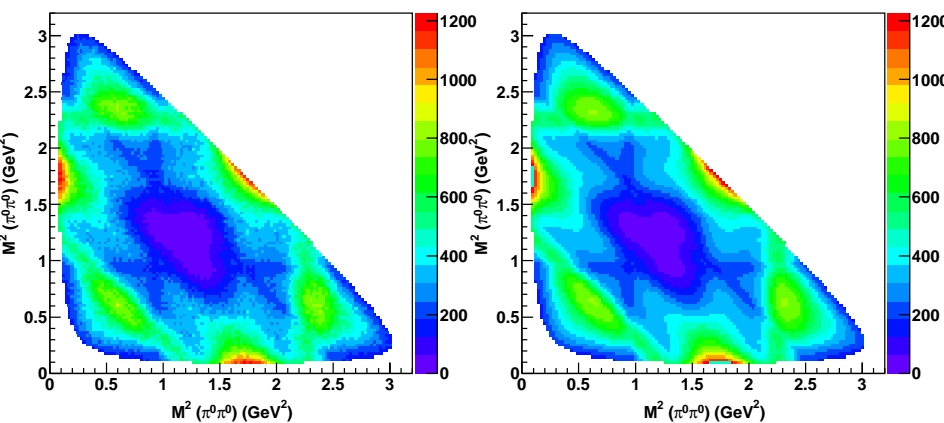


$$\bar{p}p \rightarrow \pi^0 \pi^0 \pi^0$$

( $\bar{p}p - 3\pi^0$  Liquid target)

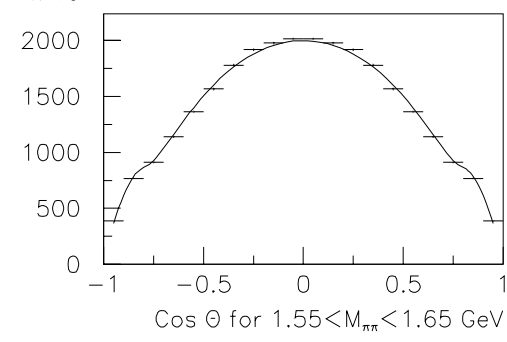
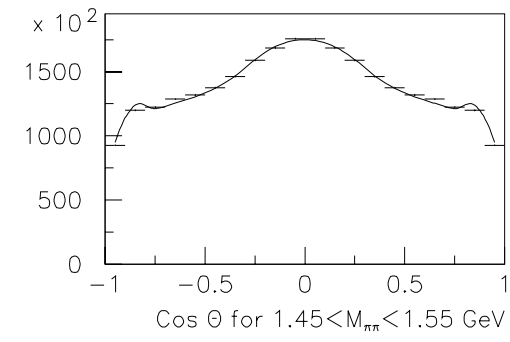
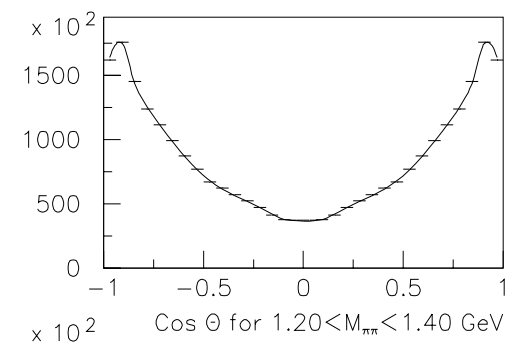
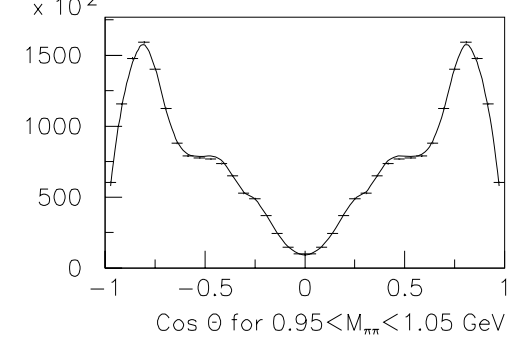
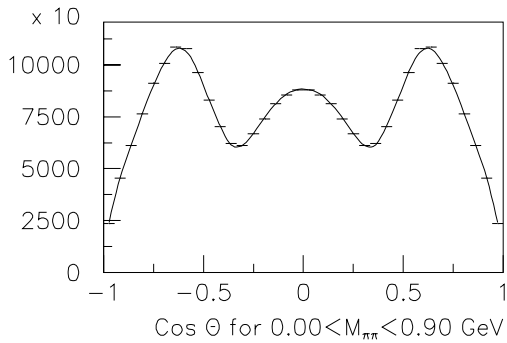
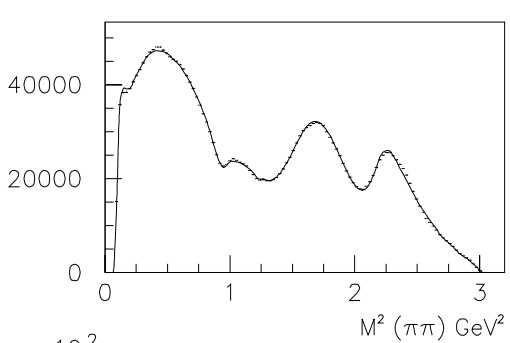
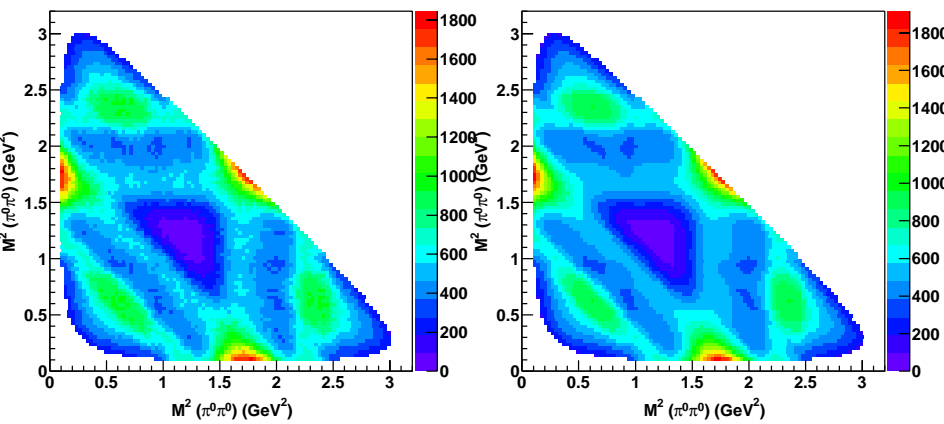
$\bar{p}p \rightarrow 3\pi^0$  in liquid hydrogen

$\bar{p}p \rightarrow 3\pi^0$  in liquid hydrogen



$\bar{p}p \rightarrow 3\pi^0$  in gaseous hydrogen

$\bar{p}p \rightarrow 3\pi^0$  in gaseous hydrogen

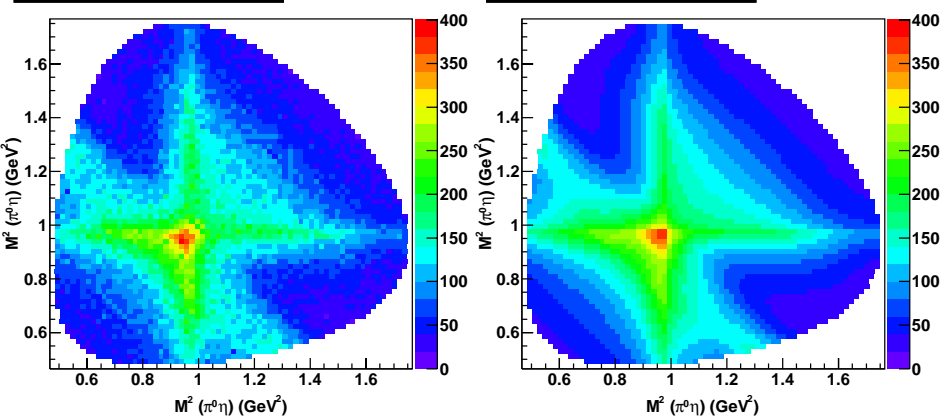


$$\bar{p}p \rightarrow \eta\eta\pi^0$$

( $p\bar{p} - \pi^0\eta\eta$  Liquid target)

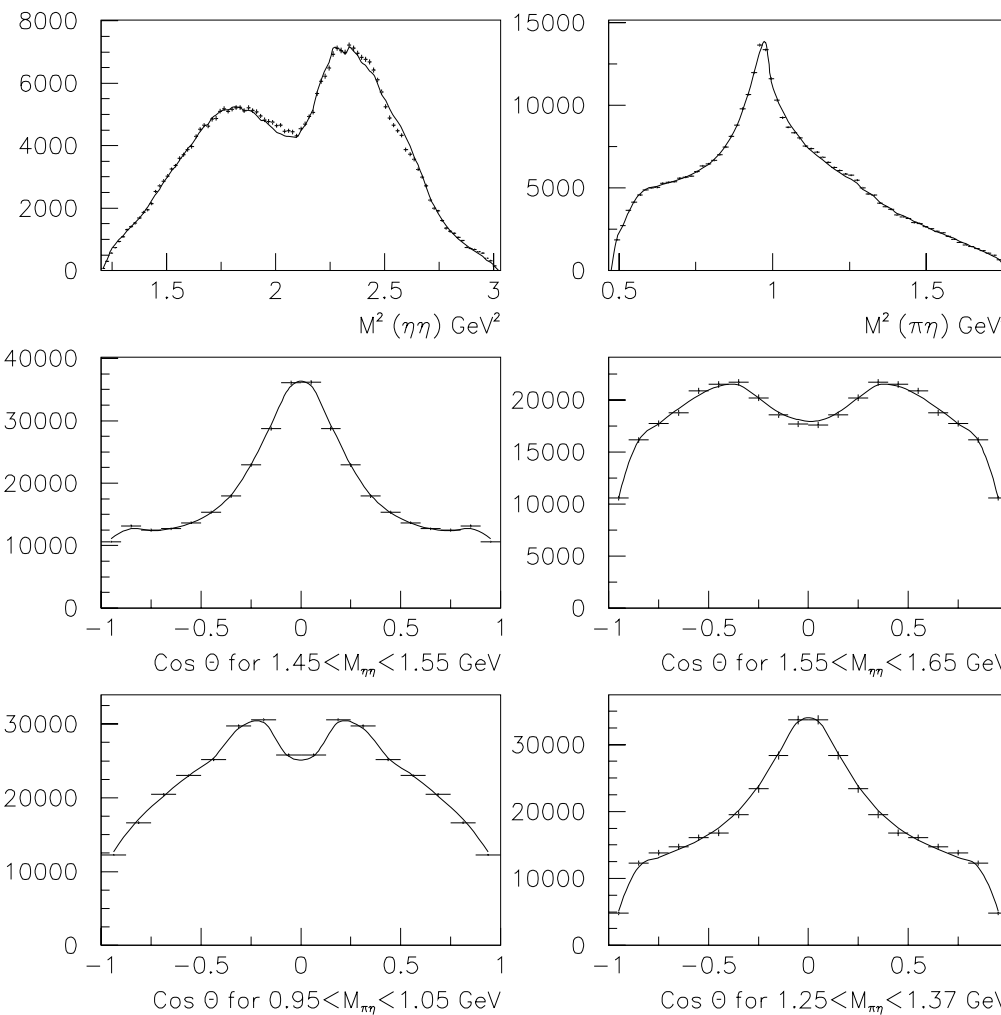
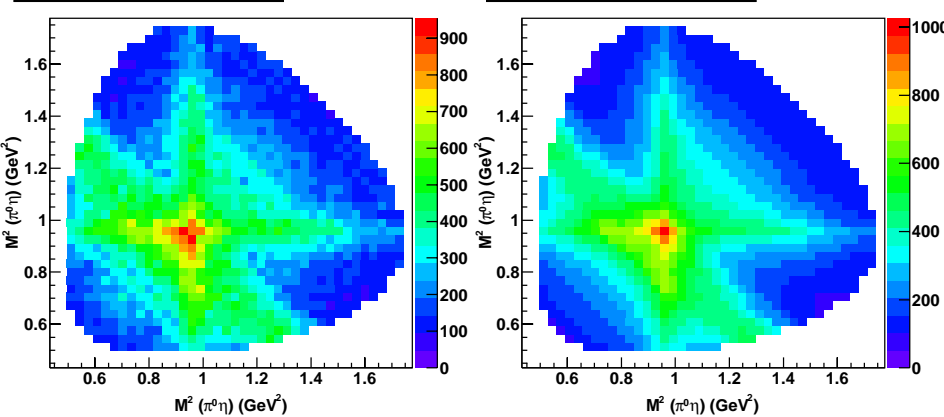
$\bar{p}p \rightarrow \pi^0\eta\eta$  in liquid hydrogen

$\bar{p}p \rightarrow \pi^0\eta\eta$  in liquid hydrogen



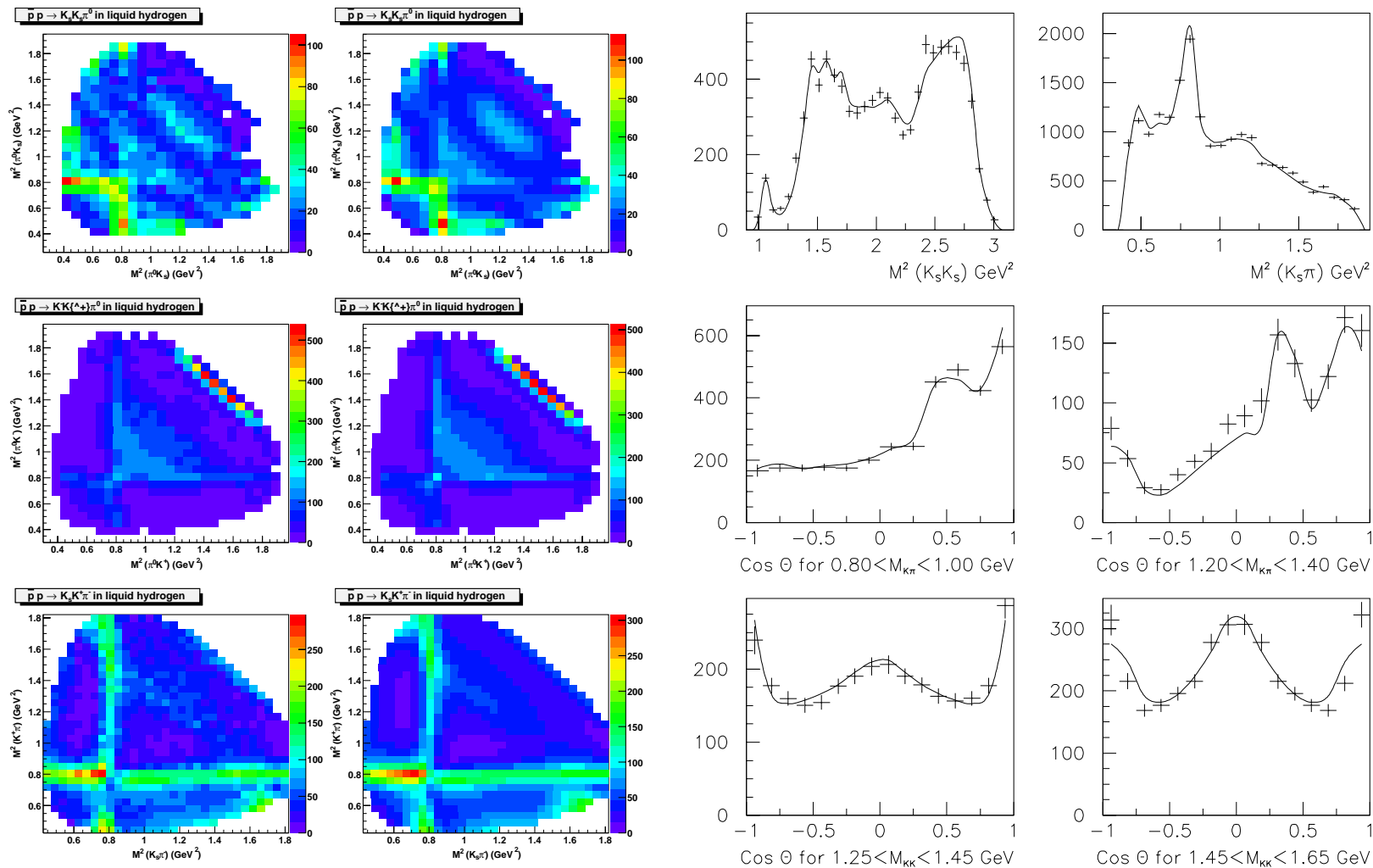
$\bar{p}p \rightarrow \pi^0\eta\eta$  in gaseous hydrogen

$\bar{p}p \rightarrow \pi^0\eta\eta$  in gaseous hydrogen



$$\bar{p}p \rightarrow K\bar{K}\pi^0$$

$K_s K_s \pi^0$  in Liquid H<sub>2</sub>



## *D*-vector approach, and *PD*-approach

Production of the weak channels, or many particle final states.

$$A_a = \hat{D}_a + [\hat{K}(\hat{I} - i\hat{\rho}\hat{K})^{-1} \hat{\rho}]_{ab} \hat{D}_b ,$$

$$D_b = \sum_m \frac{g_a^{(n)} \Lambda_n^{dec}}{M_n^2 - s} + d_b$$

The decay couplings  $\Lambda_n^{dec}$  and nonresonant transition from K-matrix channel  $b$  to final channel  $d_b$  can be complex numbers.

In the case of weak initial channel:

$$A = G + P_a [(\hat{I} - i\hat{\rho}\hat{K})^{-1} \hat{\rho}]_{ab} D_b , \quad G = \sum_m \frac{\Lambda_n \Lambda_n^{dec}}{M_n^2 - s} + c$$

All resonance couplings are the same as in P and D-vectors. The direct nonresonant  $c$  can be a complex number.

### Baryon spectroscopy

1. **K-matrix:**  $\pi N \rightarrow \pi N$ ,  $\pi N \rightarrow \eta N$ ,  $\pi N \rightarrow K\Lambda$  and  $\pi N \rightarrow K\Sigma$  reactions.
2. **P-vector:**  $\gamma N \rightarrow \pi N$ ,  $\gamma N \rightarrow \eta N$ ,  $\gamma N \rightarrow K\Lambda$  and  $\gamma N \rightarrow K\Sigma$  reactions.
3. **D-vector:**  $\pi N \rightarrow \pi\pi N$
4. **PD-approach**  $\gamma N \rightarrow \pi\pi N$ ,  $\gamma N \rightarrow \pi\eta N$

**For the description of the data see talk of A.Anisovich.**

The fitted reactions. **Recently included data sets.** **New points added**

Observable	$N_{\text{data}}$	$\frac{\chi^2}{N_{\text{data}}}$		Observable	$N_{\text{data}}$	$\frac{\chi^2}{N_{\text{data}}}$	
$\sigma(\gamma p \rightarrow p\pi^0)$	<b>1106</b>	<b>1.27</b>	<b>CB-ELSA</b>	$\sigma(\gamma p \rightarrow p\pi^0)$	<b>861</b>	<b>1.74</b>	<b>GRAAL</b>
$\sigma(\frac{3}{2} - \frac{1}{2})(p\pi^0)$	<b>140</b>	<b>1.41</b>	<b>A2GDH</b>	$\Sigma(\gamma p \rightarrow p\pi^0)$	<b>1492</b>	<b>3.38</b>	<b>SAID</b>
$P(\gamma p \rightarrow p\pi^0)$	<b>607</b>	<b>3.16</b>	<b>SAID</b>	$\Gamma(\gamma p \rightarrow p\pi^0)$	<b>389</b>	<b>4.01</b>	<b>SAID</b>
$H(\gamma p \rightarrow p\pi^0)$	<b>71</b>	<b>1.92</b>	<b>SAID</b>	$G(\gamma p \rightarrow p\pi^0)$	<b>75</b>	<b>2.58</b>	<b>SAID</b>
$Ox(\gamma p \rightarrow p\pi^0)$	<b>7</b>	<b>1.01</b>	<b>SAID</b>	$Oz(\gamma p \rightarrow p\pi^0)$	<b>7</b>	<b>0.38</b>	<b>SAID</b>
$\sigma(\gamma p \rightarrow n\pi^+)$	<b>1583</b>	<b>1.87</b>	<b>SAID</b>	$\sigma(\gamma p \rightarrow n\pi^+)$	<b>408</b>	<b>2.09</b>	<b>A2GDH</b>
$\Sigma(\gamma p \rightarrow n\pi^+)$	<b>899</b>	<b>4.23</b>	<b>SAID</b>	$\sigma(\frac{3}{2} - \frac{1}{2})(n\pi^+)$	<b>231</b>	<b>2.49</b>	<b>A2GDH</b>
$P(\gamma p \rightarrow n\pi^+)$	<b>252</b>	<b>3.90</b>	<b>SAID</b>	$\Gamma(\gamma p \rightarrow n\pi^+)$	<b>661</b>	<b>3.66</b>	<b>SAID</b>
$H(\gamma p \rightarrow p\pi^0)$	<b>71</b>	<b>1.92</b>	<b>SAID</b>	$G(\gamma p \rightarrow p\pi^0)$	<b>75</b>	<b>2.58</b>	<b>SAID</b>
$S_{11}(\pi N \rightarrow \pi N)$	<b>126</b>	<b>1.40</b>	<b>SAID</b>	$P_{11}(\pi N \rightarrow \pi N)$	<b>110</b>	<b>2.24</b>	<b>SAID</b>
$P_{13}(\pi N \rightarrow \pi N)$	<b>108</b>	<b>2.57</b>	<b>SAID</b>	$P_{33}(\pi N \rightarrow \pi N)$	<b>130</b>	<b>4.56</b>	<b>SAID</b>
$D_{33}(\pi N \rightarrow \pi N)$	<b>136</b>	<b>4.51</b>	<b>SAID</b>	$D_{13}(\pi N \rightarrow \pi N)$	<b>106</b>	<b>5.06</b>	<b>SAID</b>
$\sigma(\gamma p \rightarrow p\eta)$	<b>667</b>	<b>0.92</b>	<b>CB-ELSA</b>	$\sigma(\gamma p \rightarrow p\eta)$	<b>100</b>	<b>2.72</b>	<b>TAPS</b>
$\Sigma(\gamma p \rightarrow p\eta)$	<b>51</b>	<b>2.06</b>	<b>GRAAL 98</b>	$\Sigma(\gamma p \rightarrow p\eta)$	<b>100</b>	<b>2.01</b>	<b>GRAAL 04</b>
$T(\gamma p \rightarrow p\eta)$	<b>50</b>	<b>1.52</b>	<b>Phoenixis</b>	$\sigma(\pi^- p \rightarrow n\eta)$	<b>288</b>	<b>2.76</b>	<b>CBALL+Richards</b>

The fitted reactions. **Recently included data sets.**

Observable	$N_{\text{data}}$	$\frac{\chi^2}{N_{\text{data}}}$		Observable	$N_{\text{data}}$	$\frac{\chi^2}{N_{\text{data}}}$	
$C_x(\gamma p \rightarrow \Lambda K^+)$	<b>160</b>	<b>1.22</b>	<b>CLAS</b>	$C_x(\gamma p \rightarrow \Sigma^0 K^+)$	<b>94</b>	<b>2.29</b>	<b>CLAS</b>
$C_z(\gamma p \rightarrow \Lambda K^+)$	<b>160</b>	<b>1.53</b>	<b>CLAS</b>	$C_z(\gamma p \rightarrow \Sigma^0 K^+)$	<b>94</b>	<b>2.19</b>	<b>CLAS</b>
$\sigma(\gamma p \rightarrow \Lambda K^+)$	<b>1377</b>	<b>1.80</b>	<b>CLAS</b>	$\sigma(\gamma p \rightarrow \Sigma^0 K^+)$	<b>1280</b>	<b>2.68</b>	<b>CLAS</b>
$P(\gamma p \rightarrow \Lambda K^+)$	<b>202</b>	<b>2.31</b>	<b>CLAS</b>	$P(\gamma p \rightarrow \Sigma^0 K^+)$	<b>95</b>	<b>1.56</b>	<b>CLAS</b>
$\Sigma(\gamma p \rightarrow \Lambda K^+)$	<b>66</b>	<b>2.70</b>	<b>GRAAL</b>	$\Sigma(\gamma p \rightarrow \Sigma^0 K^+)$	<b>42</b>	<b>0.67</b>	<b>GRAAL</b>
$\Sigma(\gamma p \rightarrow \Lambda K^+)$	<b>45</b>	<b>1.75</b>	<b>LEP</b>	$\Sigma(\gamma p \rightarrow \Sigma^0 K^+)$	<b>45</b>	<b>1.03</b>	<b>LEP</b>
$T(\gamma p \rightarrow \Lambda K^+)$	<b>66</b>	<b>2.11</b>	<b>GRAAL</b>	$\sigma(\gamma p \rightarrow \Sigma^+ K^0)$	<b>48</b>	<b>3.36</b>	<b>CLAS</b>
$Ox(\gamma p \rightarrow \Lambda K^+)$	<b>66</b>	<b>1.40</b>	<b>GRAAL</b>	$\sigma(\gamma p \rightarrow \Sigma^+ K^0)$	<b>160</b>	<b>0.95</b>	<b>CB-ELSA</b>
$Oz(\gamma p \rightarrow \Lambda K^+)$	<b>66</b>	<b>1.86</b>	<b>GRAAL</b>	$P(\gamma p \rightarrow \Sigma^+ K^0)$	<b>72</b>	<b>0.72</b>	<b>CB-ELSA</b>
$\sigma(\gamma p \rightarrow p\pi^0\pi^0)$	<b>CB-ELSA (1.4 GeV)</b>			$E(\gamma p \rightarrow p\pi^0\pi^0)$	<b>16</b>	<b>2.08</b>	<b>MAMI</b>
$\sigma(\gamma p \rightarrow p\pi^0\eta)$	<b>CB-ELSA (3.2 GeV)</b>			$\Sigma(\gamma p \rightarrow p\pi^0\eta)$	<b>180</b>	<b>2.68</b>	<b>GRAAL</b>
$\sigma(\gamma p \rightarrow p\pi^0\pi^0)$	<b>CB-ELSA (3.2 GeV)</b>			$\Sigma(\gamma p \rightarrow p\pi^0\pi^0)$	<b>128</b>	<b>0.85</b>	<b>GRAAL</b>
$\sigma(\pi^- p \rightarrow K\Lambda)$	<b>479</b>	<b>1.55</b>	<b>RAL</b>	$P(\pi^- p \rightarrow K\Lambda)$	<b>261</b>	<b>1.76</b>	<b>RAL+ANL</b>
$\sigma(\pi^+ p \rightarrow K^+\Sigma)$	<b>609</b>	<b>1.91</b>	<b>RAL</b>	$P(\pi^+ p \rightarrow K^+\Sigma)$	<b>420</b>	<b>2.74</b>	<b>RAL</b>

$L, S, N$	$\kappa_{gd}$	Resonance					Pred.
$0, \frac{1}{2}, 0$	$\frac{1}{2}$	$N(940)$				input:	<b>0.94</b>
$0, \frac{3}{2}, 0$	<b>0</b>	$\Delta(1232)$					<b>1.27</b>
$0, \frac{1}{2}, 1$	$\frac{1}{2}$	$N(1440)$					<b>1.40</b>
$1, \frac{1}{2}, 0$	$\frac{1}{4}$	$N(1535)$	$N(1520)$				<b>1.53</b>
$1, \frac{3}{2}, 0$	<b>0</b>	$N(1650)$	$N(1700)$	$N(1675)$			<b>1.64</b>
$1, \frac{1}{2}, 0$	<b>0</b>	$\Delta(1620)$	$\Delta(1700)$		$L, S, N=0, \frac{3}{2}, 1:$	$\Delta(1600)$	<b>1.64</b>
$2, \frac{1}{2}, 0$	$\frac{1}{2}$	$N(1720)$	$N(1680)$		$L, S, N=0, \frac{1}{2}, 2:$	$N(1710)$	<b>1.72</b>
$1, \frac{1}{2}, 1$	$\frac{1}{4}$	$N(????)$	$N(1875)$				<b>1.82</b>
$1, \frac{3}{2}, 1$	<b>0</b>	$\Delta(1900)$	$\Delta(1940)$	$\Delta(1930)$			<b>1.92</b>
$2, \frac{3}{2}, 0$	<b>0</b>	$\Delta(1910)$	$\Delta(1920)$	$\Delta(1905)$	$\Delta(1950)$		<b>1.92</b>
$2, \frac{3}{2}, 0$	<b>0</b>	$N(1880)$	$N(1900)$	$N(1990)$	$N(2000)$		<b>1.92</b>
$0, \frac{1}{2}, 3$	$\frac{1}{2}$	$N(2100)$					<b>2.03</b>
$3, \frac{1}{2}, 0$	$\frac{1}{4}$	$N(2070)$	$N(2190)$	$L, S, N=1, \frac{1}{2}, 2:$	$N(2080)$	$N(2090)$	<b>2.12</b>
$3, \frac{3}{2}, 0$	<b>0</b>	$N(2200)$	$N(2250)$	$L, S, N=1, \frac{1}{2}, 2:$	$\Delta(2223)$	$\Delta(2200)$	<b>2.20</b>
$4, \frac{1}{2}, 0$	$\frac{1}{2}$	$N(2220)$					<b>2.27</b>
$4, \frac{3}{2}, 0$	<b>0</b>	$\Delta(2390)$	$\Delta(2300)$	$\Delta(2420)$	$ L, N=3, 1:$	$\Delta(2400)$	<b>2.43</b>
$5, \frac{1}{2}, 0$	$\frac{1}{4}$	$N(2600)$			$ $	$\Delta(2350)$	<b>2.57</b>



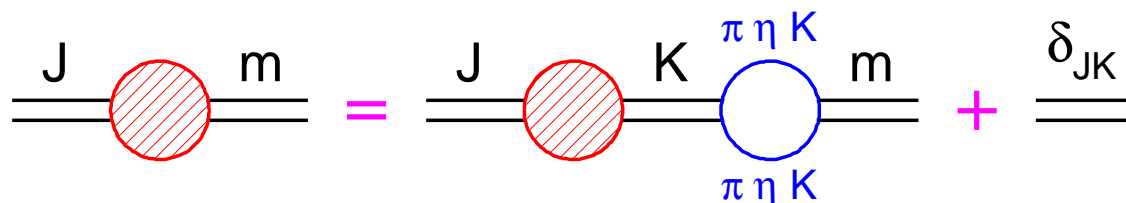
1. **K-matrix approach satisfies the unitarity condition. It takes into account right-hand side singularities of the amplitude: threshold singularities (cuts) and pole singularities.**
2. **The P-vector, D-vector and PD-methods allow us to perform of analysis of many reactions simultaneously.**
3. **This approach is fast enough to perform the analysis of modern high statistical data and reliably extract leading singularities of the amplitude. This is impossible with most approaches which solve directly integral (e.g. Bethe-Salpeter) equations.**

**However:**

**This approach does not take into account left-hand side singularities (connected with the real part of loop diagrams) and therefore is not fully reliable at very low energies or in presence of strong thresholds**

## N/D based analysis of the data

In the case of resonance contributions only we have factorization and Bethe-Salpeter equation can be easily solved:



$$A_{jm} = A_{jk} \sum_{\alpha} B_{\alpha}^{km}(s) \frac{1}{M_m - s} + \frac{\delta_{jm}}{M_j^2 - s} \quad B_{\alpha}^{km}(s) = \int_{4m_j^2}^{\infty} \frac{ds'}{\pi} \frac{g_{\alpha}^{(k)}(s') \rho(s') g_{\alpha}^{(m)}(s')}{s' - s - i0}$$

$$\hat{A} = \hat{\kappa}(I - \hat{B}\hat{\kappa})^{-1} \quad \kappa_{ij} = \frac{\delta_{ij}}{M_i^2 - s} \quad B^{ij} = \sum_{\alpha} B_{\alpha}^{km}(s)$$

For non-resonant contributions: there is no factorization and the amplitude can have a complicated energy dependence. **However in majority of K-matrix analysis the nonresonant contributions are constant or have a simple energy dependence.**

Non-factorization can be taken into account by introduction of two transitions with fixed left and right vertices.

**Parameterization of  $P_{13}$  wave: 3 resonances 8 channels, 4 non-resonant contributions**  
 $\pi N \rightarrow \pi N, \pi N \rightarrow \eta N, \pi N \rightarrow K\Sigma, \pi N \rightarrow \Delta\pi$ . It corresponds to  **$8 \times 8$  channel K-matrix** and  **$5 \times 5$  N/D-matrix**.

In many cases (fixed form-factor or subtraction procedure) the real part can be calculated in advance. If only imaginary part is taken into account the method is completely corresponds to the K-matrix approach.

$$A_{ab} = \sum_{ij} \text{Diagram 1} \quad P_b = \sum_{ij} \text{Diagram 2}$$

1. This approach satisfies analyticity and two body unitarity conditions.
2. It takes left-hand side singularities into account.
3. The approach is suitable for the analysis of high statistic data in combined analysis of many reactions.

## SUMMARY

1. To extract resonance properties from modern experimental data a combined partial wave analysis of many reactions is needed.
2. Such analysis should take into account interference of different states with same quantum numbers and satisfy (at least two body) unitarity condition.
3. The influence of left-hand side singularities should be taken into account or at least controlled.
4. The data from different experiments should be easily accessible after publication.

There are at least four groups which perform combined analysis of the data in baryon spectroscopy. **But there are no PWA groups which analyze a comparable number of data sets in meson spectroscopy.**