”D-matrix (N/D inspired) method for the partial wave analysis of experimental data”

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1 Unitarity and analyticity of the scattering amplitude

The scattering matrix is unitary.

$$S_{if} =< i|S|f > \qquad\qquad \sum_m |m >< m| = 1$$

Then

$$\sum_m P_{mi} = 1 = \sum_m | < m|S|i > |^2 = \sum_m < i|S^+|m >< m|S|i >=< i|S^+S|i >$$

This must be true for any state $|i >$ **we have:**

$$S^+S = I = SS^+$$

In two body case:

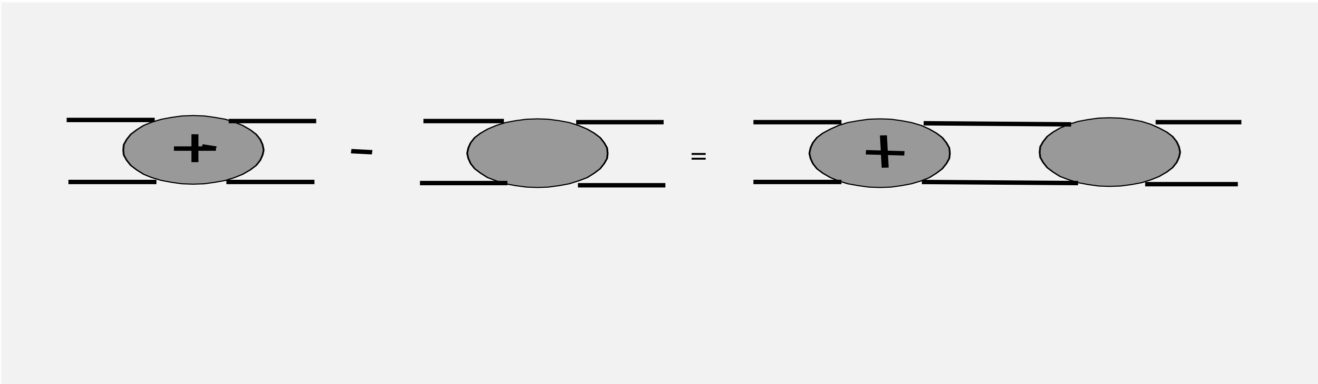
$$\int \prod_{i=1}^{2} \frac{d^3q_i}{2(2\pi)^3 q_{0i}} < p_1'p_2'|S|q_1q_2 >< q_1q_2|S^+|p_1p_2 >=< p_1'p_2'|p_1p_2 >$$

Let us introduce the scattering amplitude

$$S_{fi} = I + 2\pi\, i\, \delta^4(P_f - P_i) A_{fi}(s,t)$$

Then for amplitude A:

$$i(< f|A^+|i > - < f|A|i >) = (2\pi)^4 \sum_n \delta^4(p_n - p_i) < f|A|n >< n|A^+|i >$$



For identical initial and final states one has:

$$2Im < i|A|i > = (2\pi)^4 \sum_n \delta^4(p_n - p_i) < i|A|n >< n|A^+|i >$$

2 Partial wave amplitudes

Let us consider $2 \to 2$ scattering amplitude of two spinless particles. In the c.m.s. of two particle system:

$$A(s,t) = 16\pi \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(z) A(s)$$

If particles has the equal mass m then:

$$t = (p_1' - p_1)^2 = 2m^2 - 2p_{01}' p_{01} + 2|\vec{p}'||\vec{p}|z$$

$$p_{01}' = p_{01} = \frac{\sqrt{s}}{2} \qquad |\vec{p}'| = |\vec{p}| = \sqrt{\frac{s-4m^2}{4s}}$$

Then:

$$A(s) = \frac{1}{16\pi} \int_{-1}^{1} \frac{dz}{2} A(s,t) P_L(z) \qquad s = 4m^2 - \frac{2t}{1-z}$$

The unitarity equation is valid for every partial wave and has a very simple form.

$$16\pi \sum_l (2l+1)(A^l_{if}(s) - A^{l+}_{if}(s))P_l(z) = i\frac{16|\vec{q}|}{\sqrt{s}} \int\limits_0^{2\pi} d\Phi \int\limits_{-1}^{1} dz'$$

$$\Big[\sum_{l'}(2l'+1)A^{l'}_{in}(s)P_{l'}(z')\sum_{l''}(2l''+1)A^{l''+}_{nf}(s)P_{l''}(z'')\Big]$$

where:

$$z = \cos\Theta \qquad z' = \cos\Theta' \qquad z'' = \cos\Theta''$$

$$\cos\Theta' = \cos\Theta\cos\Theta'' + \sin\Theta\sin\Theta''\sin\Phi$$

Using orthogonality condition of the Legendre functions we obtain:

$$A^l_{if}(s) - A^{l+}_{if} = \frac{4i|\vec{q}|}{\sqrt{s}} A^l_{in}(s) A^{l*}_{nf}(s)$$

In the matrix form:

$$Im\hat{A}^l = \hat{\rho}(s)\hat{A}^l(s)\hat{A}^{l*}(s)$$

Here $\hat{\rho}(s)$ is the diagonal matrix of the phase volumes:

$$\hat{\rho}(s) = \begin{pmatrix} \rho_1(s) & 0 & \dots \\ 0 & \rho_2(s) & \dots \\ \dots & \dots & \dots \end{pmatrix} \qquad \rho_j(s) = \frac{2|\vec{q}|}{\sqrt{s}}$$

It is also useful to introduce a partial wave S-matrix:

$$\hat{S} = I + 2i\hat{\rho}(s)\hat{A}(s)$$

Then the unitarity condition reads:

$$\hat{S}^+\hat{S} = \hat{S}\hat{S}^+ = 1$$

For one channel case:

$$S = e^{2i\delta} \qquad A = \frac{e^{2i\delta} - 1}{2i\rho(s)}$$

3 Singularities of the scattering amplitude

Analyticity means that:

1) Amplitude is a Lorentz scalar, and can be written as function of scalars.

2) Amplitude has only singularities which are demanded by the unitarity condition. It means also that if there is no singularities inside a integration counter, then:

$$\int\limits_C T(s)\frac{ds}{\pi} = 0$$

Threshold singularities in this case are defined by the phase volume functions. The lowest one (in energy scale) is the two body threshold singularity.

$$d\Phi(P;k_1,k_2) = \frac{d^3k_1}{(2\pi)^3 2k_{10}}\frac{d^3k_2}{(2\pi)^3 2k_{20}}(2\pi)^4\delta^{(4)}(P-k_1-k_2)$$

and:

$$\rho(k) = \frac{1}{2}\int d\Phi(P;k_1,k_2) \qquad\qquad \rho(k) = \frac{1}{16\pi}\sqrt{\frac{s-4m^2}{s}}$$

If particles interact via exchange of a particle with mass μ then in t-channel $A(s,t)$ has a pole at $t=\mu^2$.

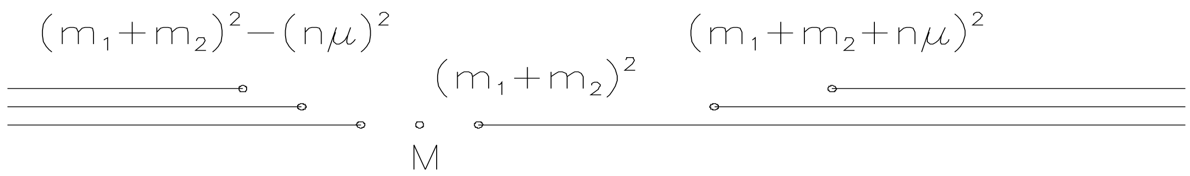
$$A(s) = \int_{-1}^{1} \frac{dz}{2} A(s,t) P_L(z) \qquad\qquad s = 4m^2 - \frac{2t}{1-z}$$

Left-hand side singularity defined by the pole singularity in t-channel:

$$z = -1 \qquad\qquad s = 4m^2 - \mu^2,$$

$$z = +1 \qquad\qquad s \to -\infty$$

Then we obtain the following picture of the singularities:



4 K-matrix representation of the scattering amplitude

The unitarity condition for the partial wave amplitude is:

$$SS^{+} = I \qquad\qquad S = I + 2i\hat{\rho}(s)\hat{A}(s)$$

Then in the case of a Breit-Wigner resonance we have:

$$A(s) = \frac{M\Gamma}{M^2 - s - i\rho(s)M\Gamma}$$

How to construct an amplitude in the case of two Breit-Wigner states? The sum of two Breit-Wigner states clearly violates unitarity condition:

$$A(s) = \frac{M_1\Gamma_1}{M_1^2 - s - i\rho(s)M_1\Gamma_1} + \frac{M_2\Gamma_2}{M_2^2 - s - i\rho(s)M_2\Gamma_2} =$$

$$\frac{M_1\Gamma_1(M_2^2 - s) + M_2\Gamma_2(M_1^2 - s) - 2i\rho(s)M_1M_2\Gamma_1\Gamma_2}{(M_1^2 - s)(M_2^2 - s) - i\rho(s)\big(M_1\Gamma_1(M_2^2 - s) + M_2\Gamma_2(M_1^2 - s) - i\rho(s)M_1M_2\Gamma_1\Gamma_2\big)}$$

One way to fulfill the unitarity condition is to introduce an approach which conserved unitarity from the beginning. Let us write:

$$S = \frac{I + i\hat{\rho}\hat{K}}{I - i\hat{\rho}\hat{K}} = I + 2i\hat{\rho}A(s), \qquad A(s) = \hat{K}(I - i\hat{\rho}\hat{K})^{-1}$$

Where $\hat{K}$ is a real matrix.

Can one obtain the K-matrix from basic principles only?

1. **The amplitude is symmetrical for transition between final states, thus K-matrix must be also symmetrical.**
2. **One should not use divergent functions as well as poles of more then first order.**
3. **The amplitude must have pole singularities not more then first order.**

Let us construct two channel one pole K-matrix. The expression like

$$K = \begin{pmatrix} \frac{G_1}{M^2-s} & \frac{G_2}{M^2-s} \\ \frac{G_2}{M^2-s} & \frac{G_3}{M^2-s} \end{pmatrix}$$

The amplitude has only first order poles if

$$G_1 G_3 = G_2^2$$

Thus we obtain factorization property of the K-matrix:

$$K_{ab} = \left(\sum_\alpha \frac{g_a^{(\alpha)} g_b^{(\alpha)}}{M_\alpha^2 - s} + f_{ab} \right), \qquad f_{ab} = f_{ba}$$

5 Property of the K-matrix amplitude in special cases

5.1 Two channel one pole K-matrix: Flatté parameterization

The amplitude for transition between two channels described by one pole K-matrix has the following form:

$$A = K(I - i\rho(s)K)^{-1} \qquad\qquad K = \begin{pmatrix} \frac{g_1 g_1}{M^2 - s} & \frac{g_1 g_2}{M^2 - s} \\ \frac{g_2 g_1}{M^2 - s} & \frac{g_2 g_2}{M^2 - s} \end{pmatrix}$$

And:

$$\begin{aligned} A_{11} &= \frac{g_1^2}{M^2 - s - i\rho_1(s)g_1^2 - i\rho_2(s)g_2^2} \\ A_{12} &= \frac{g_1 g_2}{M^2 - s - i\rho_1(s)g_1^2 - i\rho_2(s)g_2^2} \\ A_{22} &= \frac{g_2 g_2}{M^2 - s - i\rho_1(s)g_1^2 - i\rho_2(s)g_2^2}. \end{aligned}$$

1) K-matrix approach satisfies the unitarity condition. It takes into account right-hand side singularities of the amplitude: threshold singularities (cuts) and pole singularities

2) K-matrix satisfies the analyticity condition. The amplitude has a property that counter integral equal to zero if it has no singularities inside the counter.

3) K-matrix has the direct connection with field theory approach ...

However:

This approach does not take into account right-hand cut singularities (connected with the real part of loop diagrams) and therefore is not fully reliable at very low energies.

Complete overlapping of resonances: effect of accumulation of resonance width

Here we consider an example which describes a situation with mixing of completely overlapping resonances. This example demonstrates the effect of width accumulation by one of resonances.

Let us consider one channel case and a resonance situated far from threshold. Then we can neglect the s-dependence of phase space factor. Then:

$$M_1^2 = M_R^2 - iM_R\Gamma_1 \,, \qquad M_2^2 = M_R^2 - iM_R\Gamma_2 \,,$$

In this case we obtain following formula for pole position of the amplitude:

$$M_{A,B}^2 = \frac{1}{2}(M_1^2 + M_2^2) \pm \sqrt{\frac{1}{4}(-iM_R^2\Gamma_1 + iM_R^2\Gamma_2)^2 + \left(iM_R\sqrt{\Gamma_1\Gamma_2}\right)^2} =$$

$$= \begin{cases} M_R^2 - iM_R(\Gamma_1 + \Gamma_2) \\ M_R^2 \end{cases}$$

We see, that as result of mixing, one state accumulates the widths of the both initial states, $\Gamma_A = \Gamma_1 + \Gamma_2$, while another one transforms into a stable particle, $\Gamma_B = 0$.

Let us consider the example when $Re\ M_1^2$ and $Re\ M_2^2$ are different but $M_1\Gamma_1 = M_2\Gamma_2$, namely:

$$M_1^2 = M_{R1}^2 - iM\Gamma\,, \qquad M_2^2 = M_{R2}^2 - iM\Gamma\,,$$

Then:

$$M_{A,B}^2 = \frac{1}{2}(M_{R1}^2 + M_{R2}^2) - iM\Gamma \pm \sqrt{\frac{1}{4}(M_{R1}^2 - M_{R2}^2)^2 - M^2\Gamma^2}.$$

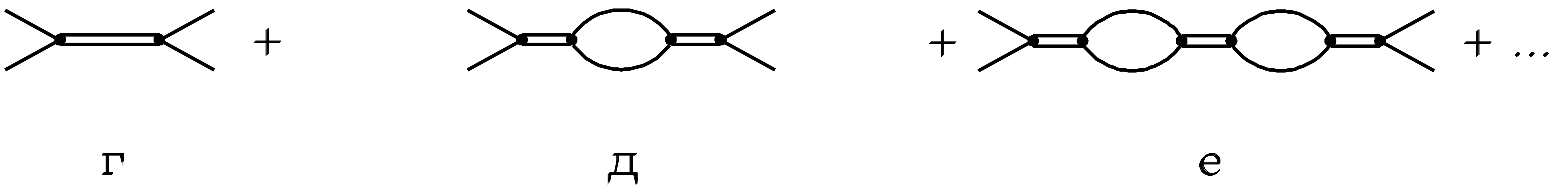
This equation allows to see dynamics of pole positions with increase of Γ. At $2M\Gamma \ll (M_{R1}^2 - M_{R2}^2)$ (that corresponds to suppressed mixing), we have two well spaced poles. Within increasing of Γ, the poles are shifted along the real axis each to other. At $2M\Gamma = M_{R1}^2 - M_{R2}^2$, the pole positions coincide:

$$M_{A,B}^2 = \frac{1}{2}(M_{R1}^2 + M_{R2}^2) - iM\Gamma$$

With the following increasing of Γ, the poles are moving along the imaginary axis. As a result we have to poles, one above another: at $(M_{R1}^2 - M_{R2}^2) \ll 2M\Gamma$, one state is nearly stable while the width of another resonance is close to 2Γ.

Diagram approach to the calculation of the scattering amplitude

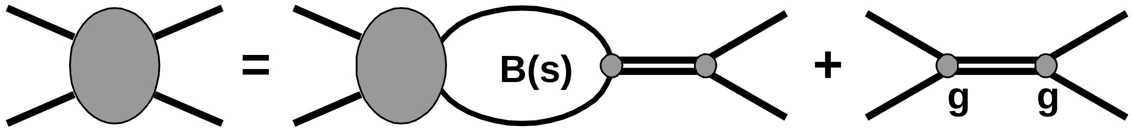
Let us consider a state which is produced from two interacting particles, then propagates and decays into same two particles in the final state. The amplitude for such process can be represented as a sum of the diagrams:



The amplitude can be found as direct sum of the diagrams:

$$A = \frac{1}{M_0^2 - s} \frac{1}{1 - \frac{B(s)}{M_0^2 - s}}$$

This amplitude also can be found by solving the following equation:



$$A = A\frac{B(s)}{M_0^2 - s} + \frac{g^2}{M_0^2 - s}$$

$$A = \frac{g^2}{M_0^2 - s}\frac{1}{1 - \frac{B(s)}{M_0^2 - s}} = \frac{g^2}{M_0^2 - s - B(s)}$$

Here M_0 is a bare mass of the state and $B(s)$ is the two body loop diagram:

$$B^F = \int \frac{d^4k}{i(2\pi)^4}\frac{g^2}{(m^2 - k^2)(m^2 - (P-k)^2)},$$

Let us assume that the vertex (coupling) g has no singularities and is a smooth function in the physical region. The imaginary part of the loop diagram appears at the energy $s > 4m^2$. The discontinuity on the cut can be calculated by substituting propagators by delta functions (Mandelstam-Cutkosky rule):

$$(m^2 - k^2)^{-1}(m^2 - (P-k)^2)^{-1} \rightarrow (-2\pi)^2 i\; \delta(m^2 - k^2)\Theta(k_0)$$
$$\delta(m^2 - (P-k)^2)\Theta(P_0 - k_0)$$

Then in c.m.s. $P = (\sqrt{s}, \vec{0})$:

$$\int \frac{d^4k^2}{4\pi^2}\delta(m^2 - k^2)\delta(m^2 - (P-k)^2) = \int \frac{d^3k}{4\pi^2}\delta(2\sqrt{s}\sqrt{m^2 + |\vec{k}|^2} - s) =$$
$$\int \frac{d\vec{k}^2}{8\pi}\frac{|\vec{k}|}{\sqrt{s}}\delta(\vec{k}^2 - \frac{s-4m^2}{4}) = \frac{1}{16\pi}\sqrt{\frac{s-4m^2}{s}} = \rho(s)$$

Then loop diagram can be rewritten in the dispersion representation:

$$B(s) = \int\limits_{4m^2}^{\infty} \frac{ds'}{\pi}\frac{g(s')\rho(s')g(s')}{s'-s-i0} = Re\int\limits_{4m^2}^{\infty}\frac{ds'}{\pi}\frac{g(s')\rho(s')g(s')}{s'-s} - i\rho(s)g^2(s)$$

Let us calculate analytically the loop diagram assuming that g is a constant.

First we need a renormalization:

$$B(s) = B(M^2) + (s - M^2) \int\limits_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{g^2}{(s'-s)(s'-M^2)} \sqrt{\frac{s'-4m^2}{s'}}$$

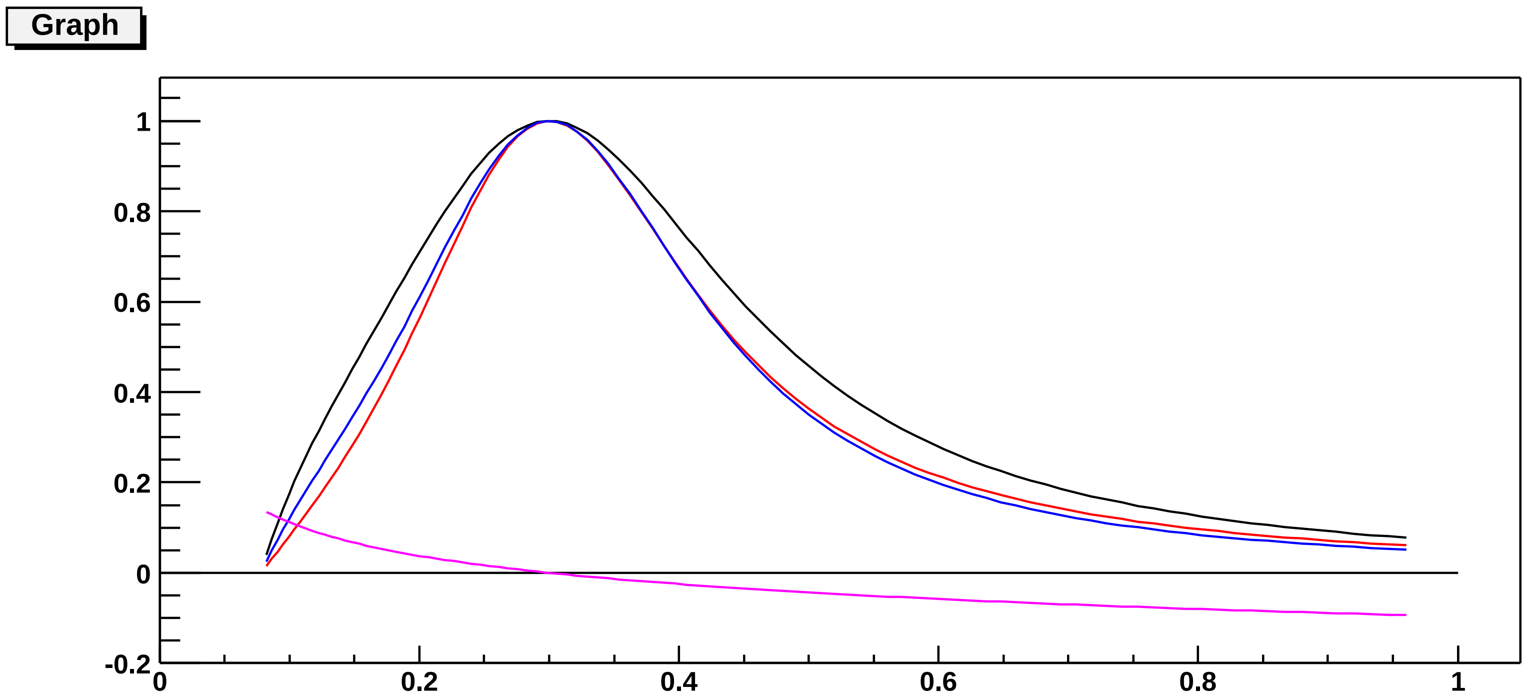
Such integral is equal to:

$$B(s) = ReB(M^2) + \frac{g^2}{\pi}[\rho(s)\; ln\frac{1-\rho(s)}{1+\rho(s)} - \rho(M^2)\; ln\frac{1-\rho(M^2)}{1+\rho(M^2)}] + i\rho(s)g^2$$

At $s \to 0$:

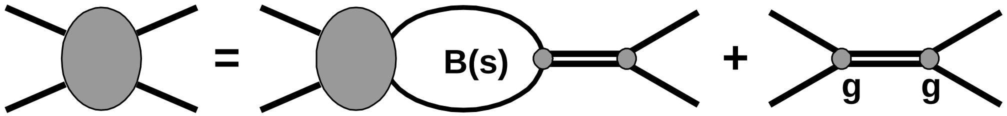
$$i\rho(s) = i\sqrt{\frac{s-4m^2}{s}} \to -\infty$$

$$i\rho(s)(1. - \frac{i}{\pi} ln\frac{1-\rho(s)}{1+\rho(s)}) = i\sqrt{\frac{s-4m^2}{s}}(1 - \frac{2}{\pi} arctg\frac{4m^2-s}{s}) \to const$$



Black curve - BW amplitude, red curve - full B(s) calculation, blue curve - BW amplitude with reduced width, magenta - dispersion correction of the real part.

The K-matrix amplitude can be considered as a solution of Bethe-Salpeter equation:



$$A_{ab}(s,s) = \int\limits_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{A_{aj}(s,s')i\rho_j(s')K_{jb}(s')}{s'-s-i0} + K_{ab}(s)$$

But ... with omitted real part of loop diagrams:

$$A_{ab} = A_{aj} i\rho_j(s) K_{jb} + K_{ab} \qquad \rightarrow \qquad \hat{\mathbf{A}} = \hat{\mathbf{K}}(\mathbf{I} - \mathbf{i}\hat{\rho}\hat{\mathbf{K}})^{-\mathbf{1}}$$

Dispersion relation N/D-method and the K-matrix representation

Let us write a partial wave amplitude $A(s)$ in the form

$$A(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ function has only left-hand side singularities of the amplitude and D-function has only right-hand side singularities. Amplitude poles correspond to zeros of the D-function and the asymptotical condition is:

$$N(s) \to 0 \qquad D(s) \to 1 \qquad \text{at} \qquad s \to \infty$$

Then from unitarity condition $ImA = \rho A^+ A$ we have:

$$ImD(s) = -\rho(s)N(s) \qquad \text{at} \qquad s > 4m^2$$

$$D(s) = 1 - B(s) = 1 - \int\limits_{4m^2}^{\infty} \frac{\rho(s')N(s')}{s'-s}\frac{ds}{\pi}$$

In the simplest case when $N(s)$ is a smooth function in the physical region one can introduce a factorization

$$N(s) = G(s)G(s) \quad or \quad N(s) = \sum_\alpha G_\alpha(s)G_\alpha(s)$$

and then:

$$A = \frac{G^2}{1 - ReB(s) - i\rho G^2(s)}$$

In this case poles appear at $Re\ B(s) = 1$. This is a so called dynamical pole. In the region of this pole we can expand the real part of $B(s)$:

$$ReB(s) = 1 + ReB'(M^2)(s - M^2)$$

And obtain:

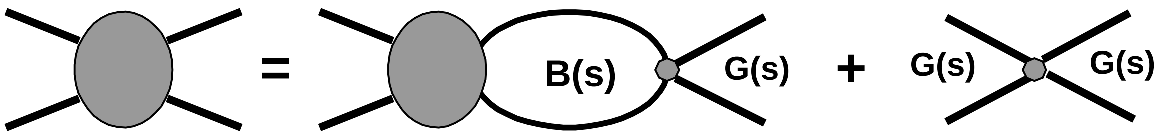
$$A = \frac{G^2/ReB'(M^2)}{M^2 - s - i\rho G^2(s)/ReB'(M^2)}$$

a Breit-Wigner expression with

$$M\Gamma = \rho(M^2)G^2(M^2)/ReB'(M^2)$$

The function can be written as a sum of the dispersion diagrams:

$A \qquad G(s)G(s) \qquad G(s)B(s)G(s) \qquad G(s)B(s)B(s)G(s)$



$$A = A\ B(s) + G^2(s) \qquad\qquad A = \frac{G^2}{1 - ReB(s) - i\rho G^2(s)}$$

In the two channel case:

$$A = \hat{N}\hat{D}^{-1} \qquad \hat{N} = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}$$

$$Im\hat{D} = -\hat{\rho}\hat{N} \qquad \hat{D} = \begin{pmatrix} 1 - B_{11} & -B_{12} \\ -B_{21} & 1 - B_{22} \end{pmatrix}$$

$$A = N\hat{D}^{-1} = \frac{1}{detD} \begin{pmatrix} 1 - B_{22} & B_{12} \\ B_{21} & 1 - B_{11} \end{pmatrix}$$

$$\begin{aligned} A_{12} &= N_{11}B_{12} + N_{12} - N_{12}B_{11} \\ A_{21} &= N_{22}B_{21} + N_{21} - N_{21}B_{22} \end{aligned}$$

$$B_{ij}(s) = \int\limits_{s_i}^{\infty} \frac{ds'}{\pi} \frac{\rho_i(s')N_{ij}(s')}{s' - s - i0}$$

The diagram approach to the multi channel amplitudes

$$N_{ij}(s,s') = \sum_k N_{ij}^k(s,s') = \sum_n G_{ij}^k(s) G_{ji}^k(s')$$

Let us introduce the block a_{ijk}^m, for transition from i to k with fixed last interaction $N_{jk}^m(s)$ and wothout final vertex $G_{ji}^k(s)$:

$$A_{ik} = \sum_{mj} a_{ijk}^m G_{jk}^m.$$

The a_{ijk}^m obeys the following equation:

$$a_{ijk}^m = \sum_{n,l} a_{ilj}^n B_{ljk}^{nm} + \delta_{ij} G_{kj}^m,$$

where one loop diagram B_{ljk}^{nm} is equal to:

$$B_{ljk}^{nm}(s) = \int \frac{ds'}{\pi} \, \frac{G_{lj}^n(s') G_{jk}^m(s') \rho_j(s)}{s'-s},$$

In the matrix form:

$$\hat{a} = \begin{pmatrix} a^1_{111} & a^1_{211} & \cdots \\ a^2_{111} & a^2_{211} & \cdots \\ \cdots & \cdots & \cdots \\ a^1_{121} & a^1_{221} & \cdots \\ a^2_{121} & a^2_{221} & \cdots \\ \cdots & \cdots & \cdots \\ a^1_{112} & a^1_{212} & \cdots \\ a^2_{112} & a^2_{212} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} \qquad \hat{g} = \begin{pmatrix} G^1_{11} & 0 & \cdots \\ G^2_{11} & 0 & \cdots \\ \cdots & \cdots & \cdots \\ 0 & G^1_{21} & \cdots \\ 0 & G^2_{21} & \cdots \\ \cdots & \cdots & \cdots \\ G^1_{12} & 0 & \cdots \\ G^2_{12} & 0 & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}.$$

And the equation is:

$$(I - \hat{B})\hat{a} = \hat{g},$$

Here $\hat{B}$ is the matrix of functions B^{mn}_{ljk} with dimension $M \times M$, where M is the number of combinations for j, k, n. If the number of channels is equal to N, then matrix $\hat{a}$ è $\hat{g}$

have dimensions $M \times N$. Then

$$\hat{A} = \hat{g}'\hat{a} = \hat{g}'(I - \hat{B})^{-1}\hat{g},$$

where $\hat{g}'$ is matrix $N \times M$ with elements $\delta_{ik}G^m_{jk}$. Here i is the line number and column number runs through all combinations j, k, m:

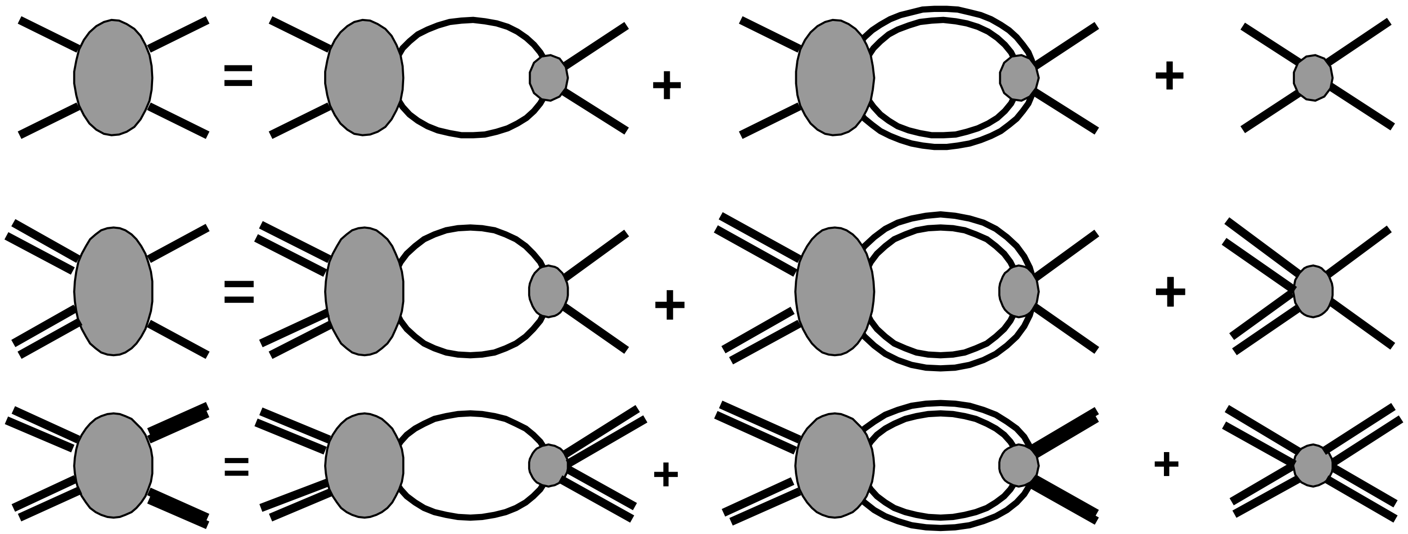
$$\hat{g} = \begin{pmatrix} G^1_{11} & G^2_{11} & \cdots & G^1_{21} & G^2_{21} & \cdots & 0 & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & G^1_{12} & G^2_{12} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

The unitarity condition

$$Im\hat{B} = \hat{g}\hat{\rho}\hat{g}',$$

5.2 K-matrix poles and $q\bar{q}$ states

Let us consider the amplitude in the case of two final states. In this case we have a set of diagram equations:



$$\hat{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad \hat{B} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \qquad \hat{G} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

$$\hat{A} = \hat{A}\,\hat{B} + \hat{G} \qquad\qquad \hat{A} = \hat{G}(I - \hat{B})^{-1}$$

For A_{11} we obtain:

$$A_{11} = \frac{G_{11}(I - B_{22}) - G_{12}B_{21}}{(I - B_{11})(I - B_{22}) - B_{12}B_{21}} = \frac{G_{11} - G_{12}T_{21}}{(I - B_{11}) - B_{12}T_{21}}$$

Where T_{21} is equal to:

$$T_{21} = (I - B_{22})^{-1}B_{21} = B_{21} + B_{22}B_{21} + B_{22}B_{22}B_{21} + B_{22}B_{22}B_{22}B_{21} + \ldots$$

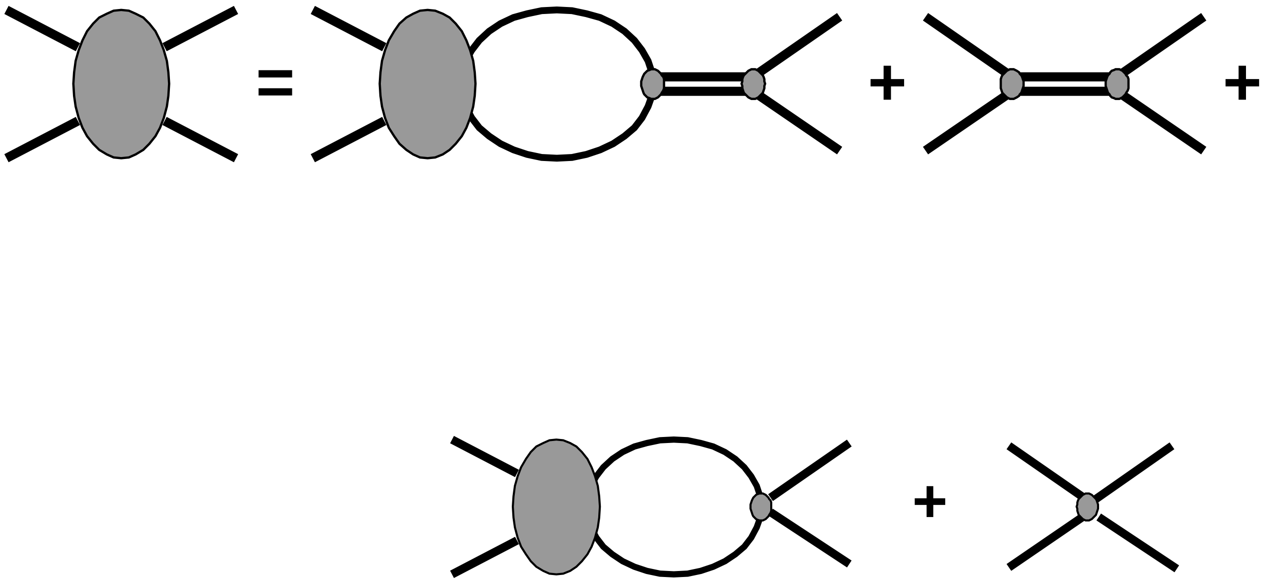
Then for the amplitude $1 \to 1$ we obtain the following equation:

$$A_{11} = A_{11}B_{11} + A_{11}B_{12}T_{21} + G_{11} + G_{12}T_{21}$$

If, for example, the channel 2 is the quark-antiquark channel then:

$$G_{12}T_{21} = \sum_{\alpha} \frac{g_{\alpha}^2}{M_{0\alpha}^2 - s}$$

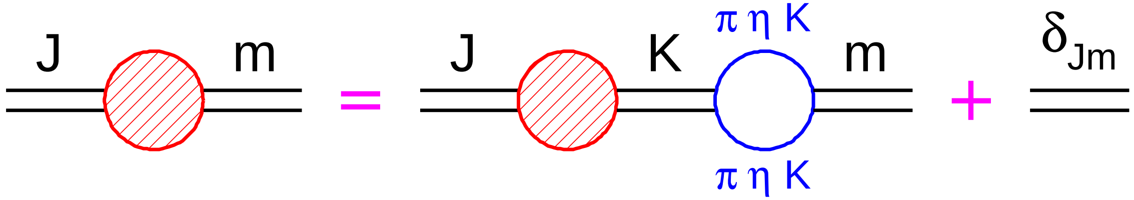
If real part is smooth we obtain the standard K-matrix expression:



$$A = K(I - i\rho K)^{-1} \qquad\qquad K = \sum_{\alpha} \frac{g_{\alpha}^2}{M_{\alpha}^2 - s} + G_{11}$$

N/D based (D-matrix) analysis of the data

In the case of resonance contributions only we have factorization and Bethe-Salpeter equation can be easily solved:



$$D_{jm} = D_{jk} \sum_{\alpha} B_{\alpha}^{km}(s) \frac{1}{M_m - s} + \frac{\delta_{jm}}{M_j^2 - s} \qquad \hat{D} = \hat{\kappa}(I - \hat{B}\hat{\kappa})^{-1}$$

$$\hat{\kappa} = diag\left(\frac{1}{M_1^2 - s}, \frac{1}{M_2^2 - s}, \ldots, \frac{1}{M_N^2 - s}, R_1, R_2 \ldots\right)$$

$$\hat{B}_{ij} = \sum_{\alpha} B_{\alpha}^{ij} = \sum_{\alpha} \int \frac{ds'}{\pi} \frac{g_{\alpha}^{(R)i} \rho_{\alpha}(s', m_{1\alpha}, m_{2\alpha}) g_{\alpha}^{(L)j}}{s' - s - i0}$$

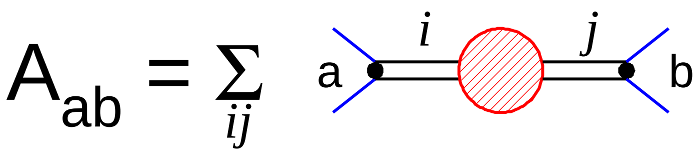
For non-resonant transitions from one channel (e.g. $\pi\pi$):

$$g_i^{L(N+1)} R_1 g_j^{R(N+1)} + g_i^{L(N+2)} R_2 g_j^{R(N+2)}$$

where N is the number of pole terms. The non-zero left and right vertices:

$$g_j^{L(N+1)} = f_{1j} \frac{1\ \textbf{GeV}^2 + s_0}{s + s_0} \qquad g_1^{R(N+1)} = 1 \qquad R_1 = 1\,,$$

$$g_1^{L(N+2)} = 1 \qquad g_{j>1}^{R(N+2)} = f_{1j} \frac{1\ \textbf{GeV}^2 + s_0}{s + s_0} \qquad R_2 = 1$$



$$A_{ab} = \sum_{\alpha,\beta} g_a^{R(\alpha)} \kappa_{\alpha\alpha} D_{\alpha\beta} g_b^{L(\beta)}\ .$$

In the present fits we calculate the elements of the B^{ij}_α using one subtraction taken at the channel threshold $M_\alpha = (m_{1\alpha} + m_{2\alpha})$:

$$B^{ij}_\alpha(s) = B^{ij}_\alpha(M^2_\alpha) + (s - M^2_\alpha) \int\limits_{m^2_a}^{\infty} \frac{ds'}{\pi} \frac{g^{(R)i}_\alpha \rho_\alpha(s', m_{1\alpha}, m_{2\alpha}) g^{(L)j}_\alpha}{(s' - s - i0)(s' - M^2_\alpha)} .$$

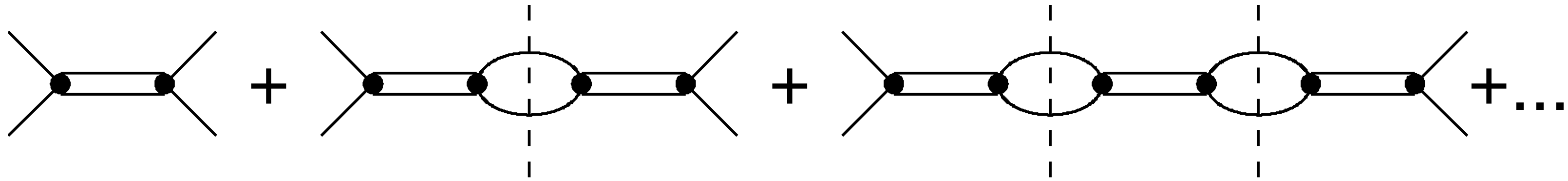
In this case the expression for elements of the $\hat{B}$ matrix can be rewritten as:

$$B^{ij}_\alpha(s) = g^{(R)i}_a \left(b^\alpha + (s - M^2_\alpha) \int\limits_{m^2_a}^{\infty} \frac{ds'}{\pi} \frac{\rho_\alpha(s', m_{1\alpha}, m_{2\alpha})}{(s' - s - i0)(s' - M^2_\alpha)} \right) g^{(L)j}_\beta = g^{(R)i}_a B_\alpha g^{(L)j}_\beta$$

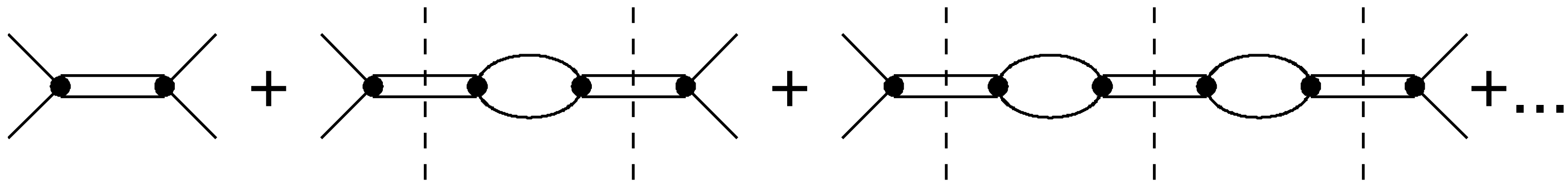
and D-matrix method equivalent to the K-matrix method with loop diagram with real part taken into account:

$$A = \hat{K}(I - \hat{B}\hat{K})^{-1} \qquad\qquad B_{\alpha\beta} = \delta_{\alpha\beta} B_\alpha$$

K-matrix approach

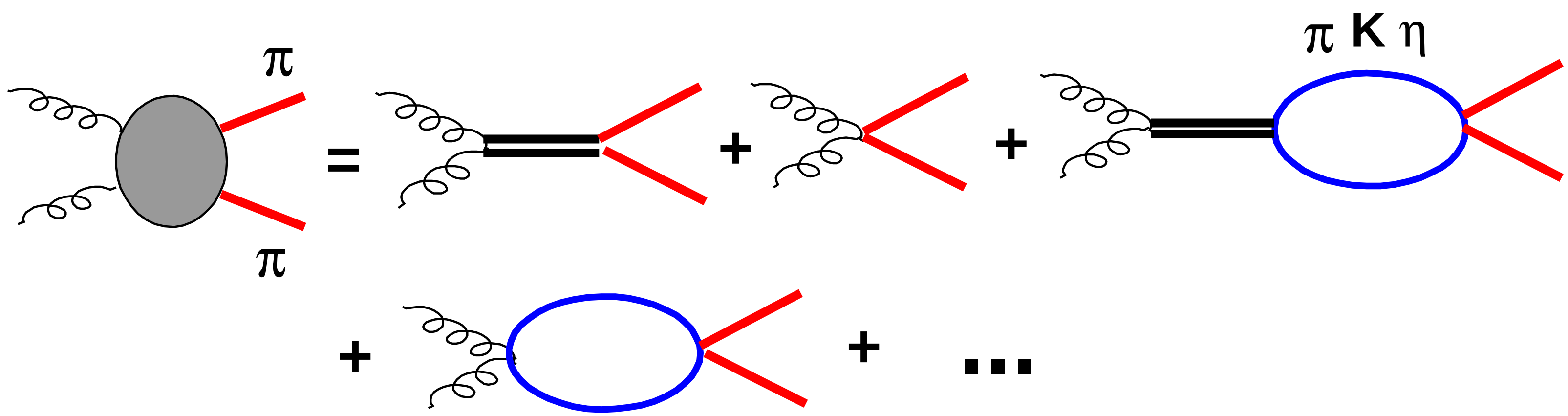


D-matrix approach



P-vector approach

Let us consider photoproduction of two pions. This case is different from the $\pi\pi$ scattering by the first interaction:



The first interaction can be the direct production of K-matrix poles or nonresonant production: $\gamma\gamma \to \pi\pi$, $\gamma\gamma \to K\bar{K}$ and so on.

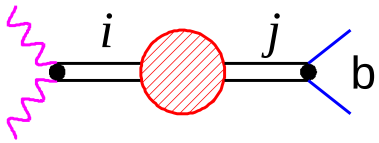
The vector of the initial interactions can be constructed as:

$$P = \begin{pmatrix} \sum\limits_m \frac{\Lambda_n g_1^n}{M_n^2 - s} + f_1 \\ \sum\limits_m \frac{\Lambda_n g_2^n}{M_n^2 - s} + f_2 \\ \ldots\ldots \end{pmatrix}$$

$$A_k = P_j (I - i\rho K)^{-1}_{jk}$$

Here m counts number of K-matrix poles, and indices $1, 2, 3 \ldots$ counts channels

$$A_b = \sum_{ij}$$



(for example $\pi\pi, \quad K\bar{K}, \quad \eta\eta \quad \ldots$).

$$A_b = \sum_{\alpha,\beta} \tilde{P}^{(\alpha)} D_{\alpha\beta} g_b^{L(\beta)} \qquad \tilde{P} = \left(\frac{\Lambda_1}{M_1^2 - s}, \frac{\Lambda_2}{M_1^2 - s}, \ldots, \frac{\Lambda_n}{M_1^2 - s}, F_1 \ldots \right)$$

F-vector approach, and PD-approach

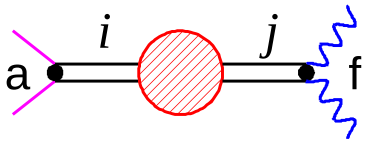
Production of the weak channels, or many particle final states.

$$A_{af} \;=\; \hat{F}_{af} + [\hat{K}(\hat{I} \;-\; i\hat{\rho}\hat{K})^{-1}\, i\hat{\rho}]_{ab}\hat{F}_{bf} \;,$$

$$\tilde{F}_{bf} = \sum_{m} \frac{g_b^{(n)} \Lambda_n^{dec}}{M_n^2 - s} + d_{bf}$$

The decay couplings Λ_n^{dec} and nonresonant transition from K-matrix channel b to final channel d_{bf} can be complex numbers.

$$A_{af} = \sum_{ij}$$



$$A_{af} = \hat{F}_{af} + \sum_{\alpha,\beta} g_a^{R(\alpha)} \kappa_{\alpha\alpha} D_{\alpha\beta} \tilde{F}_{\beta f} \qquad \tilde{F} = \left(\Lambda_1^{dec}, \Lambda_2^{dec} \ldots, \Lambda_n^{dec}, \frac{d_{bf}}{R_b} \ldots \right)$$

In the case of weak initial channel:

$$A \;=\; G + P_a[(\hat{I} \;-\; i\hat{\rho}\hat{K})^{-1}\,\hat{\rho}]_{ab}D_b\;, \qquad G = \sum_m \frac{\Lambda_n \Lambda_n^{dec}}{M_n^2 - s} + c$$

All resonance couplings are the same as in P and D-vectors. The direct nonresonant c can be a complex number.

$$A = \hat{+} \sum_{\alpha,\beta} g_a^{R(\alpha)} \kappa_{\alpha\alpha} D_{\alpha\beta} \tilde{F}_{\beta f} \qquad \tilde{F} = \left(\Lambda_1^{dec}, \Lambda_2^{dec} \ldots, \Lambda_n^{dec}, \frac{d_{bf}}{R_b} \ldots \right)$$

Phase volumes

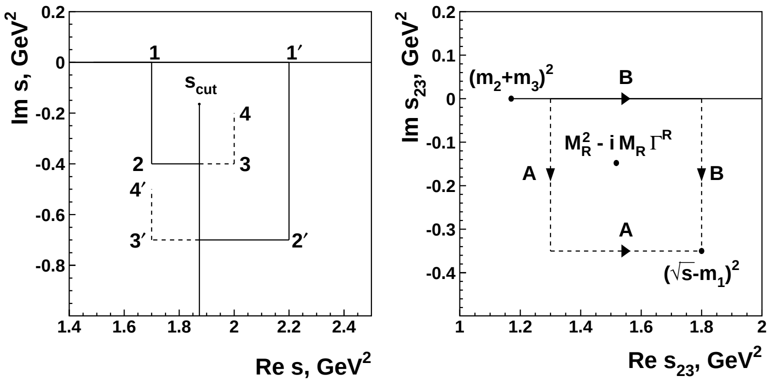
Two body phase volume:

$$\rho(s,m_1,m_2)=\frac{2k}{s}=\frac{\sqrt{(s-(m_1+m_2)^2)(s-(m_1-m_2)^2)}}{s}$$

Three body phase volume:

$$\rho_3(s)=\int\limits_{(m_2+m_3)^2}^{(\sqrt{s}-m_1)^2}\frac{ds_{23}}{\pi}\,\frac{\rho(s,\sqrt{s_{23}},m_1)\,M_R\Gamma^R_{tot}}{(M_R^2-s_{23})^2+(M_R\Gamma^R_{tot})^2}\ ,$$

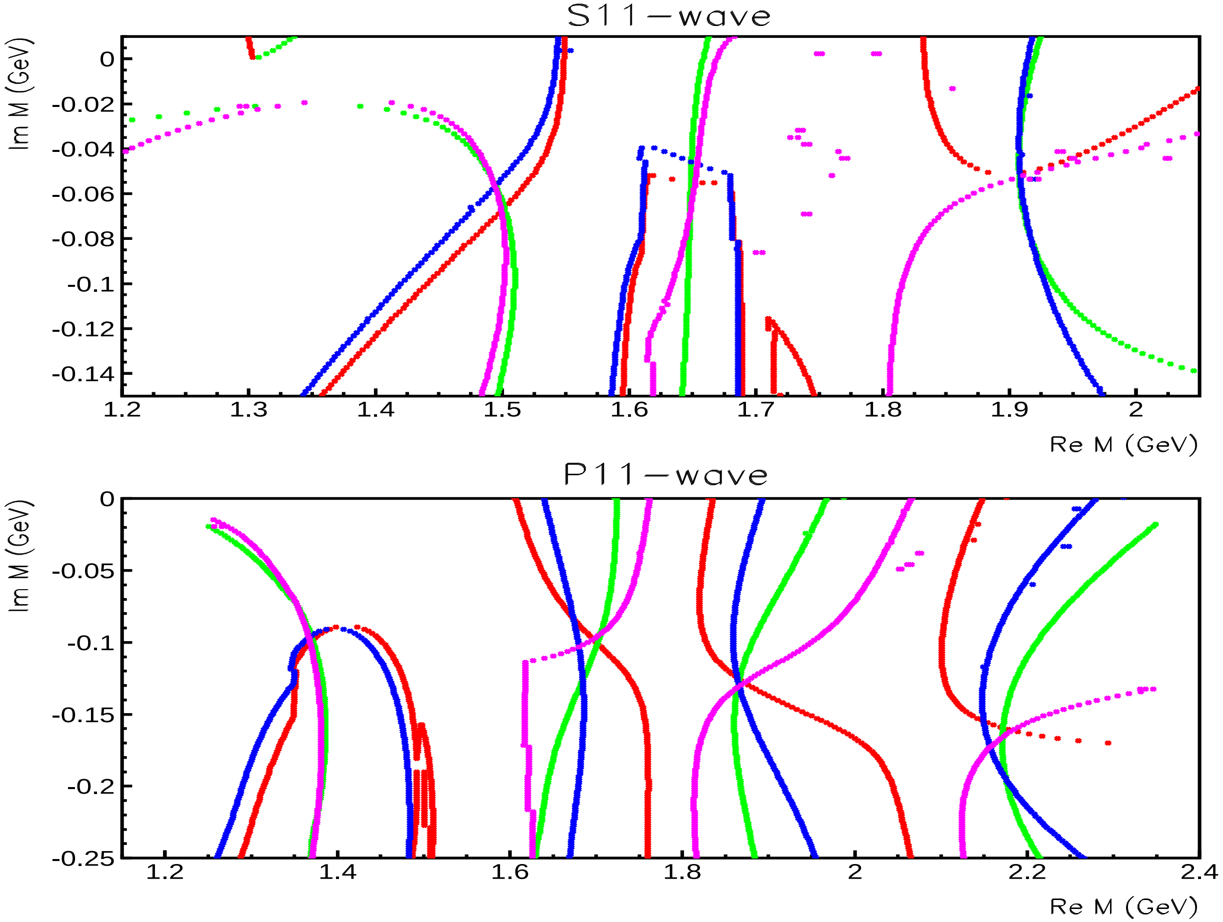
$$M_R\Gamma^R_{tot}=\rho(s_{23},m_2,m_3)g^2(s_{23})\ ,$$



Pole parameters of the S_{11} states

	$N(1535)S_{11}$		$N(1650)S_{11}$		$N(1890)S_{11}$	
	K-matrix	**D-matrix**	**K-matrix**	**D-matrix**	**K-matrix**	**D-matrix**
M_{pole}	1501$\pm$4	1494	1647$\pm$6	1651	1900$\pm$15	1905
Γ_{pole}	134$\pm$11	116	103$\pm$8	95	90^{+30}_{-15}	106
Elastic residue	31$\pm$4	25	24$\pm$3	23	1$\pm$1	1.5
Phase	-(29$\pm$5)$^\circ$	-38°	-(75$\pm$12)$^\circ$	-62°	–	–
Res$_{\pi N\to N\eta}$	28$\pm$3	25	15$\pm$3	15	4$\pm$2	5
Phase	-(76$\pm$8)$^\circ$	-69°	(132$\pm$10)$^\circ$	140	(40$\pm$20)$^\circ$	42°
Res$_{\pi N\to \Delta\pi}$	7$\pm$4	4	11$\pm$3	12	–	–
Phase	(147$\pm$17)$^\circ$	157°	-(30$\pm$20)$^\circ$	-40	–	–
$A^{1/2}$ (GeV$^{-\frac{1}{2}}$)	0.116$\pm$0.010	0.107	0.033$\pm$0.007	0.029	0.012$\pm$0.006	0.010
Phase	(7$\pm$6)$^\circ$	1°	-(9$\pm$15)$^\circ$	0°	120$\pm$50°	150°

Pole position in the mass complex plane for S_{11} and P_{11}.



Trajectories for

Re $det(I - i\rho K) = 0$

Im $det(I - i\rho K) = 0$

Re $det(I - B\kappa) = 0$

Im $det(I - B\kappa) = 0$