Introduction to Bonn-Gatchina partial wave analysis method

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Partial wave amplitude:

transition amplitude with fixed initial and final states

Quantum numbers: mesons $I^G\ J^{PC}$, baryons: IJ^P, decay LS basis: $^{2S+1}L_J$

$$I_1^{G_1} J_1^{P_1C_1} + I_2^{G_2} J_2^{P_2C_2} \left(^{2S+1}L_J\right) \to I^G J^{PC} \to I_1'^{G_1'} J_1'^{P_1'C_1'} + I_2'^{G_2'} J_2'^{P_2'C_2'} \left(^{2S'+1}L_J'\right)$$

$$G = G_1 G_2 \qquad G = G_1' G_2'$$

$$P = P_1 P_2 (-1)^L \qquad P = P_1' P_2' (-1)^{L'}$$

$$|I_1 - I_2| < I < I_1 + I_2 \qquad |I_1' - I_2'| < I < I_1' + I_2'$$

$$|J_1 - J_2| < S < J_1 + J_2 \qquad |J_1' - J_2'| < S' < J_1' + J_2'$$

$$|S - L| < J < S + L \qquad |S' - L'| < J < S' + L'$$

$$A(s,t) = V_{\mu_1\ldots\mu_n}(S,L)\, P^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_n}\, V'_{\nu_1\ldots\nu_n}(S',L') A(s)$$

$n = J$ **mesons** $\qquad n = J - 1/2$ **baryons**

In momentum representation the particle with spin J ($n = J$):

$$\Psi_{\mu_1 \dots \mu_n} = \frac{1}{\sqrt{2p_0}} u_{\mu_1 \dots \mu_n} e^{ipx}$$

The spinor function $u_{\mu_1 \dots \mu_n}$ satisfies:

$$p^2 u_{\mu_1 \mu_2 \dots \mu_n} = m^2 u_{\mu_1 \mu_2 \dots \mu_n}$$

$$p_{\mu_i} u_{\mu_1 \mu_2 \dots \mu_n} = 0$$

$$g_{\mu_i \mu_j} u_{\mu_1 \mu_2 \dots \mu_n} = 0$$

$$u_{\mu_1 \dots \mu_i \dots \mu_j \dots \mu_n} = u_{\mu_1 \dots \mu_j \dots \mu_i \dots \mu_n}$$

These conditions are the main basis for the construction of the projection operators, which are defined as:

$$G^{\mu_1 \mu_2 \dots \mu_n}_{\nu_1 \nu_2 \dots \nu_n} = P^{\mu_1 \mu_2 \dots \mu_n}_{\nu_1 \nu_2 \dots \nu_n} \frac{1}{p^2 - m^2}$$

1 Boson projection operators

In momentum representation:

$$P^{\mu_1\mu_2\ldots\mu_n}_{\nu_1\nu_2\ldots\nu_n} = (-1)^n O^{\mu_1\mu_2\ldots\mu_n}_{\nu_1\nu_2\ldots\nu_n} = \sum_{i=1}^{2n+1} u^{(i)}_{\mu_1\mu_2\ldots\mu_n} u^{(i)*}_{\nu_1\nu_2\ldots\nu_n}$$

The projection operator can depends only on the total momentum and the metric tensor. For spin 0 it is a unit operator. For spin 1 the only possible combination is:

$$O^{\mu}_{\nu} = g^{\perp}_{\mu\nu} = g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}$$

The propagator for the particle with spin $S > 2$ must be constructed from the tensors $g^{\perp}_{\mu\nu}$: this is the only combination which satisfies:

$$p_\mu g^{\perp}_{\mu\nu} = 0.$$

Then for spin 2 state we obtain:

$$O^{\mu_1\mu_2}_{\nu_1\nu_2} = \frac{1}{2}(g^{\perp}_{\mu_1\nu_1} g^{\perp}_{\mu_2\nu_2} + g^{\perp}_{\mu_1\nu_2} g^{\perp}_{\mu_2\nu_1}) - \frac{1}{3} g^{\perp}_{\mu_1\mu_2} g^{\perp}_{\nu_1\nu_2}$$

Recurrent expression for the boson projector operator

$$O^{\mu_1\ldots\mu_L}_{\nu_1\ldots\nu_L} = \frac{1}{L^2}\left(\sum_{i,j=1}^{L} g^{\perp}_{\mu_i\nu_j} O^{\mu_1\ldots\mu_{i-1}\mu_{i+1}\ldots\mu_L}_{\nu_1\ldots\nu_{j-1}\nu j+1\ldots\nu_L} - \right.$$

$$\left. \frac{4}{(2L-1)(2L-3)} \sum_{i<j,k<m}^{L} g^{\perp}_{\mu_i\mu_j} g^{\perp}_{\nu_k\nu_m} O^{\mu_1\ldots\mu_{i-1}\mu_{i+1}\ldots\mu_{j-1}\mu_{j+1}\ldots\mu_L}_{\nu_1\ldots\nu_{k-1}\nu_{k+1}\ldots\nu_{m-1}\nu_{m+1}\ldots\nu_L}\right)$$

Normalization condition:

$$O^{\mu_1\ldots\mu_L}_{\nu_1\ldots\nu_L} O^{\nu_1\ldots\nu_L}_{\alpha_1\ldots\alpha_L} = O^{\mu_1\ldots\mu_L}_{\alpha_1\ldots\alpha_L}$$

Orbital momentum operator

The angular momentum operator is constructed from momenta of particles k_1, k_2 and metric tensor $g_{\mu\nu}$.

For $L=0$ this operator is a constant: $X^0 = 1$

The $L=1$ operator is a vector $X^{(1)}_\mu$, constructed from: $k_\mu = \frac{1}{2}(k_{1\mu} - k_{2\mu})$ and $P_\mu = (k_{1\mu} + k_{2\mu})$. Orthogonality:

$$\int \frac{d^4k}{4\pi} X^{(1)}_{\mu_1} X^{(0)} = \int \frac{d^4k}{4\pi} X^{(n)}_{\mu_1\ldots\mu_n} X^{(n-1)}_{\mu_2\ldots\mu_n} = \xi P_{\mu_1} = 0$$

Then:

$$X^{(1)}_\mu P_\mu = 0 \qquad\qquad X^{(n)}_{\mu_1\ldots\mu_n} P_{\mu_j} = 0$$

and:

$$X^{(1)}_\mu = k^\perp_\mu = k_\nu g^\perp_{\nu\mu}; \qquad\qquad g^\perp_{\nu\mu} = \left(g_{\nu\mu} - \frac{P_\nu P_\nu}{p^2} \right) \; ;$$

in c.m.s $k^\perp = (0, \vec{k})$

$$\int \frac{d^4k}{4\pi} X^{(n)}_{\mu_1\ldots\mu_n} X^{(n-2)}_{\mu_3\ldots\mu_n} = \beta g^{\perp}_{\mu_1\mu_2} = 0$$

The orthogonality and symmetry properties can be written as the set of following conditions:

1.	$X^{(n)}_{\mu_1\ldots\mu_i\ldots\mu_j\ldots\mu_n}$	$= X^{(n)}_{\mu_1\ldots\mu_j\ldots\mu_i\ldots\mu_n}$	**(symmetry)**
2.	$P_{\mu_i} X^{(n)}_{\mu_1\ldots\mu_i\ldots\mu_n}$	$= 0$	**(P-orthogonality)**
3.	$g_{\mu_1\mu_2} X^{(n)}_{\mu_1\mu_2\ldots\mu_n}$	$= 0$	**(tracelessness)**

For low orbital momenta:

$$X^0 = 1\,; \quad X^1_\mu = k^\perp_\mu\,; \quad X^2_{\mu\nu} = \frac{3}{2} \quad k^\perp_\mu k^\perp_\nu - \frac{1}{3} k^2_\perp g^\perp_{\mu\nu} \quad ;$$

$$X^3_{\mu\nu\alpha} = \frac{5}{2} \quad k^\perp_\mu k^\perp_\nu k^\perp_\alpha - \frac{k^2_\perp}{5} \quad g^\perp_{\mu\nu} k^\perp_\alpha + g^\perp_{\mu\alpha} k^\perp_\nu + g_{\nu\alpha} k^\perp_\mu \quad ,$$

Recurrent expression for the orbital momentum operators $X^{(n)}_{\mu_1\ldots\mu_n}$

$$X^{(n)}_{\mu_1\ldots\mu_n} = \frac{2n-1}{n^2}\sum_{i=1}^{n} k^\perp_{\mu_i} X^{(n-1)}_{\mu_1\ldots\mu_{i-1}\mu_{i+1}\ldots\mu_n} - \frac{2k_\perp^2}{n^2}\sum_{\substack{i,j=1\\ i<j}}^{n} g_{\mu_i\mu_j} X^{(n-2)}_{\mu_1\ldots\mu_{i-1}\mu_{i+1}\ldots\mu_{j-1}\mu_{j+1}\ldots\mu_n}$$

Taking into account the traceless property of $X^{(n)}$ we have:

$$X^{(n)}_{\mu_1\ldots\mu_n} X^{(n)}_{\mu_1\ldots\mu_n} = \alpha(n)(k_\perp^2)^n \qquad \alpha(n) = \prod_{i=1}^{n}\frac{2i-1}{i} = \frac{(2n-1)!}{n!}.$$

From the recursive procedure one can get the following expression for the operator $X^{(n)}$:

$$X^{(n)}_{\mu_1\ldots\mu_n} = \prod_{k=1}^{n}\frac{2k-1}{k}\Bigg)\ k^\perp_{\mu_1}k^\perp_{\mu_2}\ldots k^\perp_{\mu_n} - \frac{k_\perp^2}{2n-1}\ g^\perp_{\mu_1\mu_2}k^\perp_{\mu_3}\ldots k^\perp_{\mu_n} + \cdots \ +$$

$$\frac{k_\perp^4}{(2n-1)(2n-3)}\ g^\perp_{\mu_1\mu_2}g^\perp_{\mu_3\mu_4}k^\perp_{\mu_5}\cdots k_{\mu_4} + \cdots \ + \cdots \ .$$

Scattering of two spinless particles

Denote relative momenta of particles before and after interaction as q and k, correspondingly. The structure of partial–wave amplitude with angular moment $L = J$ is determined by the convolution of the operators $X^{(L)}(k)$ and $X^{(L)}(q)$:

$$A_L = BW_L(s) X^{(L)}_{\mu_1\ldots\mu_L}(k) O^{\mu_1\ldots\mu_L}_{\nu_1\ldots\nu_L} X^{(L)}_{\nu_1\ldots\nu_L}(q) = BW_L(s) X^{(L)}_{\mu_1\ldots\mu_L}(k) X^{(L)}_{\mu_1\ldots\mu_L}(q)$$

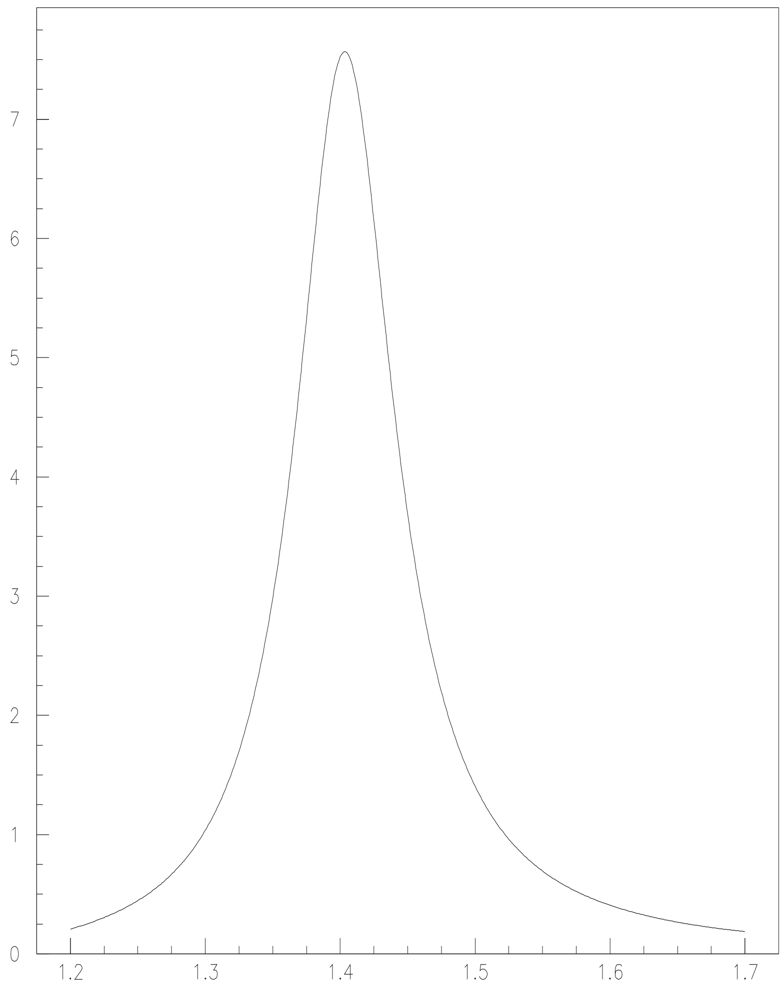
$BW_L(s)$ **is the amplitude which depends on the total energy squared only.**

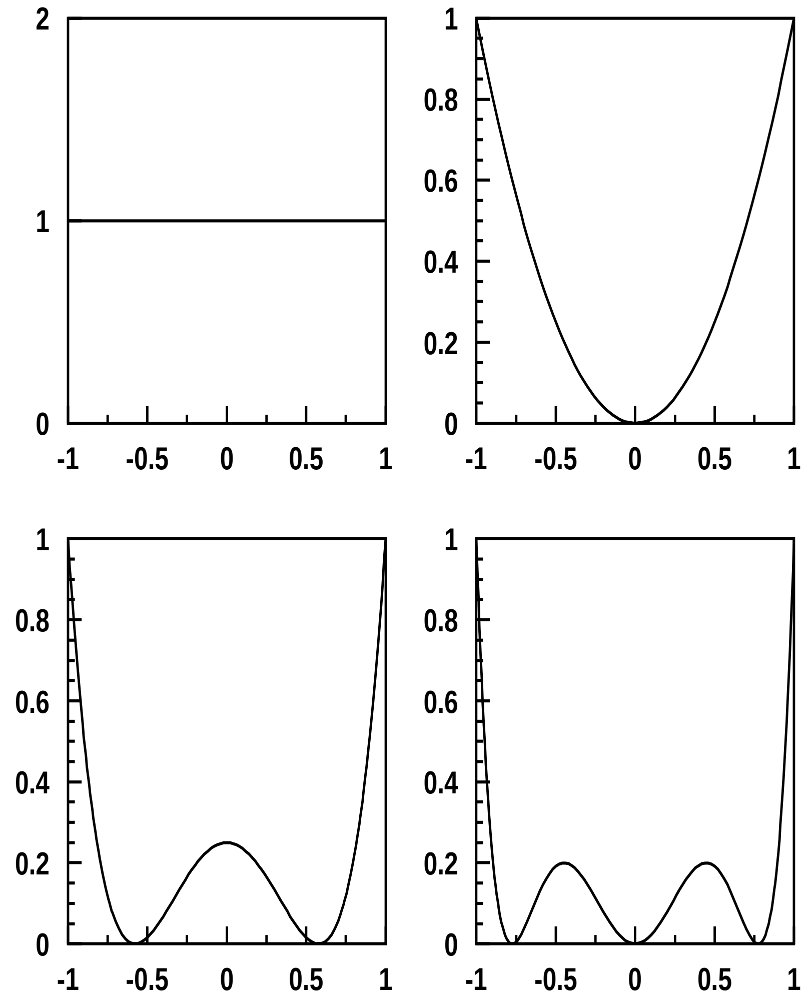
The convolution $X^{(L)}_{\mu_1\ldots\mu_L}(k) X^{(L)}_{\mu_1\ldots\mu_L}(q)$ can be written in terms of Legendre polynomials $P_L(z)$:

$$X^{(L)}_{\mu_1\ldots\mu_L}(k) X^{(L)}_{\mu_1\ldots\mu_L}(q) = \alpha_l \left(\sqrt{k^2_\perp}\sqrt{q^2_\perp}\right)^L P_L(z)\,,$$

$$z = \frac{(k^\perp q^\perp)}{\sqrt{k^2_\perp}\sqrt{q^2_\perp}} \qquad\qquad \alpha_L = \prod_{n=1}^{L} \frac{2n-1}{n}$$

Angular dependence of $|A_L|^2$ for $J = L = 0, 1, 2, 3$ states.





Structure of fermion propagator

The orthogonality condition has a different form in a fermion case:

$$\int \Psi_\mu(x)\Psi^*(x)d^4x = A\,p_\mu + B\,\gamma_\mu$$

where A and B are matrices in spinor space.

It means that we have an additional condition:

$$\gamma_\mu \Psi_\mu = 0 \qquad \gamma_\mu u_\mu = 0 \qquad \Psi_\mu = \frac{1}{\sqrt{2p_0}} u_\mu e^{ipx}$$

$$\gamma_0 = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \vec{\gamma} = \beta\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}$$

Here $\vec{\sigma}$ are 2×2 Pauly matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$u^{(i)} = \frac{1}{\sqrt{p_0+m}} \begin{pmatrix} (p_0+m)\omega^{(i)} \\ (\vec{p}\vec{\sigma})\omega^{(i)} \end{pmatrix} \qquad \omega^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \omega^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\bar{u}^{(i)} = \frac{\left((p_0+m)\omega^{(i)*}, -(\vec{p}\vec{\sigma})\omega^{(i)*}\right)}{\sqrt{p_0+m}}$$

Summing over positive energy solutions we obtain:

$$\sum_{i=1}^{2} u^{(i)}\bar{u}^{(i)} = m + \hat{p}$$

Orthogonality conditions for $J = n + \frac{1}{2}$ spinors:

$$(\hat{p} - m)u_{\mu_1\ldots\mu_n} = 0 \qquad \hat{p} = p_\mu\gamma_\mu$$

$$p_{\mu_i}u_{\mu_1\ldots\mu_n} = 0$$

$$u_{\mu_1\ldots\mu_i\ldots\mu_j\ldots\mu_n} = u_{\mu_1\ldots\mu_j\ldots\mu_i\ldots\mu_n}$$

$$g_{\mu_i\mu_j}u_{\mu_1\ldots\mu_n} = 0$$

$$\gamma_{\mu_i}u_{\mu_1\ldots\mu_n} = 0$$

These properties define structure of the fermion projection operator $P^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_n}$:

$$G^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_n} = (-1)^n \frac{m+\hat{p}}{m^2-p^2} F^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_n}$$

The boson projector operator projects any operator to one which satisfies all boson properties. It means that we can write:

$$F^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_n} = O^{\mu_1\ldots\mu_n}_{\alpha_1\ldots\alpha_n} T^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_n} O^{\beta_1\ldots\beta_n}_{\nu_1\ldots\nu_n}$$

T-operator should be constructed from the metric tensor and γ-matrices.

$$\gamma_{\alpha_i}\gamma_{\alpha_j} = \frac{1}{2} g_{\alpha_i\alpha_j} + \sigma_{\alpha_i\alpha_j}, \qquad \text{where} \qquad \sigma_{\alpha_i\alpha_j} = \frac{1}{2}(\gamma_{\alpha_i}\gamma_{\alpha_j} - \gamma_{\alpha_j}\gamma_{\alpha_i})$$

$$\gamma_{\alpha_i}\gamma_{\alpha_j} O^{\ldots\alpha_i\ldots\alpha_j\ldots}_{\beta_1\beta_2\ldots} = 0$$

Therefore, the only nonzero structure is:

$$O^{\mu_1\mu_2\ldots}_{\alpha_1\alpha_2\ldots}\gamma_{\alpha_i}\gamma_{\beta_j} O^{\beta_1\beta_2\ldots}_{\nu_1\nu_2\ldots}$$

Then:

$$F^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_n} = O^{\mu_1\ldots\mu_n}_{\alpha_1\ldots\alpha_n} T^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_n} O^{\beta_1\ldots\beta_n}_{\nu_1\ldots\nu_n}$$

$$T^{\alpha_1\ldots\alpha_L}_{\beta_1\ldots\beta_L} = \frac{L+1}{2L+1}\big(g_{\alpha_1\beta_1} - \frac{L}{L+1}\sigma_{\alpha_1\beta_1}\big)\prod_{i=2}^{L} g_{\alpha_i\beta_i}$$

$$J = 1/2 \quad P = 1$$

$$J = 3/2 \quad P^{\mu}_{\nu} = \frac{1}{2}\Big(g^{\perp}_{\mu\nu} - \gamma^{\perp}_{\mu}\gamma^{\perp}_{\nu}/3\Big) \qquad \text{where} \qquad \gamma^{\perp}_{\mu} = g^{\perp}_{\mu\nu}\gamma_{\nu}.$$

πN interaction

Pion has quantum numbers $J^{PC}=0^{-+}$, proton $1/2^+$. Then in S-wave the only state can be formed is $1/2^-$. P-wave can form two states $1/2^+$ and $3/2^+$.
In PDG review, states are defined by quantum numbers from the πN decay: $L_{2I,2J}$. For example D_{13} means $3/2^-$ N^* state.
States with $J=L-1/2$ are called '–' states ($1/2^+$, $3/2^-$,$5/2^+$,$\ldots$) and states with $J=L+1/2$ are called '+' states ($1/2^-$, $3/2^+$,$5/2^-$,$\ldots$).

For '+' states:

$$N^+_{\mu_1\ldots\mu_n}=X^{(n)}_{\mu_1\ldots\mu_n}$$

and for '-' states:

$$N^-_{\mu_1\ldots\mu_n}=i\gamma_\nu\gamma_5 X^{(n+1)}_{\nu\mu_1\ldots\mu_n}$$

$$A_{\pi N}=\bar u(k_1)N^{*\pm}_{\mu_1\ldots\mu_L}F^{\mu_1\ldots\mu_{L-1}}_{\nu_1\ldots\nu_{L-1}}(P)N^{\pm}_{\nu_1\ldots\nu_L}u(q_1)BW^{\pm}_L(s)$$

In c.m.s. of the reaction

$$A_{\pi N} = \omega^* \left[G(s,t) + H(s,t)i(\vec{\sigma}\vec{n})\right]\omega' \qquad n_i = \frac{1}{|\vec{k}||\vec{q}|}\epsilon_{ijm}k_j q_m \ ,$$

$$G(s,t) = \sum_L \left[(L{+}1)F_L^+(s) - LF_L^-(s)\right]P_L(z) \ ,$$

$$H(s,t) = \sum_L \left[F_L^+(s) + F_L^-(s)\right]P_L'(z) \ .$$

$$F_L^+ \quad = (-1)^{L+1}(|\vec{k}||\vec{q}|)^L \sqrt{\chi_i \chi_f}\ \frac{\alpha(L)}{2L{+}1} BW_L^+(s) \ ,$$

$$F_L^- \quad = (-1)^{L}(|\vec{k}||\vec{q}|)^L \sqrt{\chi_i \chi_f}\ \frac{\alpha(L)}{L} BW_L^-(s) \ .$$

$$\chi_i = m_i + k_{i0} \qquad\qquad \alpha(L) = \prod_{l=1}^{L} \frac{2l-1}{l} = \frac{(2L-1)!!}{L!} \ .$$

γN interaction

Photon has quantum numbers $J^{PC}=1^{--}$, proton $1/2^+$. Then in S-wave two states can be formed is $1/2^-$ and $3/2^-$.

Then P-wave $1/2^+, 3/2^+$ and $1/2^+, 3/2^+, 5/2^+$.

In general case: $1/2^-, 1/2^+$ are described by two amplitudes and higher states by three amplitudes.

$$
\begin{aligned}
V^{(1+)\mu}_{\alpha_1\ldots\alpha_n} &= \gamma_\mu i\gamma_5 X^{(n)}_{\alpha_1\ldots\alpha_n}\ , & V^{(1-)\mu}_{\alpha_1\ldots\alpha_n} &= \gamma_\xi\gamma_\mu X^{(n+1)}_{\xi\alpha_1\ldots\alpha_n}\ , \\
V^{(2+)\mu}_{\alpha_1\ldots\alpha_n} &= \gamma_\nu i\gamma_5 X^{(n+2)}_{\mu\nu\alpha_1\ldots\alpha_n}\ , & V^{(2-)\mu}_{\alpha_1\ldots\alpha_n} &= X^{(n+1)}_{\mu\alpha_1\ldots\alpha_n}\ , \\
V^{(3+)\mu}_{\alpha_1\ldots\alpha_n} &= \gamma_\nu i\gamma_5 X^{(n+1)}_{\nu\alpha_1\ldots\alpha_n} g^\perp_{\mu\alpha_n}\ , & V^{(3-)\mu}_{\alpha_1\ldots\alpha_n} &= X^{(n-1)}_{\alpha_2\ldots\alpha_n} g^\perp_{\alpha_1\mu}\ .
\end{aligned}
$$

Gauge invariance: $\varepsilon_\mu q_{1\mu}=0$ where q_1-photon momentum.

$$
\varepsilon_\mu V^{(2\pm)\mu}_{\alpha_1\ldots\alpha_n} = C^\pm \varepsilon_\mu V^{(3\pm)\mu}_{\alpha_1\ldots\alpha_n}
$$

where $C^\pm$ do not depend on angles.

In c.m.s. of the reaction

$$A = \sum_i u(k_1) V^{*(i\pm)\mu}_{\alpha_1\ldots\alpha_n} F^{\beta_1\ldots\beta_n}_{\alpha_1\ldots\alpha_n} N^{(\pm)}_{\beta_1\ldots\beta_n} u(q_1)\varepsilon_\mu BW^{\pm}_L(s) = \omega^* J_\mu \varepsilon_\mu \omega' \ ,$$

$$J_\mu = i\mathcal{F}_1\sigma_\mu + \mathcal{F}_2(\vec\sigma\vec q)\frac{\varepsilon_{\mu ij}\sigma_i k_j}{|\vec k||\vec q|} + i\mathcal{F}_3\frac{(\vec\sigma\vec k)}{|\vec k||\vec q|}q_\mu + i\mathcal{F}_4\frac{(\vec\sigma\vec q)}{\vec q^2}q_\mu \ .$$

$$\mathcal{F}_1(z) = \sum_{L=0}^{\infty}[LM^+_L + E^+_L]P'_{L+1}(z) + [(L+1)M^-_L + E^-_L]P'_{L-1}(z) \ ,$$

$$\mathcal{F}_2(z) = \sum_{L=1}^{\infty}[(L+1)M^+_L + LM^-_L]P'_L(z) \ ,$$

$$\mathcal{F}_3(z) = \sum_{L=1}^{\infty}[E^+_L - M^+_L]P''_{L+1}(z) + [E^-_L + M^-_L]P''_{L-1}(z) \ ,$$

$$\mathcal{F}_4(z) = \sum_{L=2}^{\infty}[M^+_L - E^+_L - M^-_L - E^-_L]P''_L(z).$$

Our amplitudes can be algebraically rewritten to multipole representation:

$$E_L = E_L^{+(1)} + E_L^{+(2)} \qquad M_L = M_L^{+(1)} + M_L^{+(2)}$$

$$\begin{aligned} E_L^{+(1)} &= (-1)^L \sqrt{\chi_i \chi_f}\, \frac{\alpha(L)}{2L+1} \frac{(|\vec{k}||\vec{q}|)^L}{L+1} BW^+(s)\ , \\ M_L^{+(1)} &= E_L^{+(1)}\ . \end{aligned}$$

$$\begin{aligned} E_L^{+(2)} &= (-1)^L \sqrt{\chi_i \chi_f}\, \frac{\alpha(L)}{2L+1} \frac{(|\vec{k}||\vec{q}|)^L}{L+1} BW^+(s)\ , \\ M_L^{+(2)} &= -\frac{E_L^{+(2)}}{L}\ . \end{aligned}$$

2 The cross section for photoproduction processes

The differential cross section for production of two or more particles has the form:

$$d\sigma = \frac{(2\pi)^4|A|^2}{4\sqrt{(k_1k_2)^2 - m_1^2m_2^2}} d\Phi_n(k_1+k_2, q_1, \ldots, q_n) \ ,$$

where k_1 and k_2 are momenta of the initial particles (nucleon and γ in the case of photoproduction) and q_i are momenta of final state particles. The $d\Phi_n(k_1+k_2, q_1, \ldots, q_n)$ is the element of the n-body phase volume given by

$$d\Phi_n(k_1+k_2, q_1, \ldots, q_n) \quad = \quad \delta^4(k_1+k_2-\sum_{i=1}^{n} q_i) \prod_{i=1}^{n} \frac{d^3q_i}{(2\pi)^3 2q_{0i}} \ .$$

The photoproduction amplitude can be written as

$$A = \varepsilon_\mu \bar{u}_i A_\mu u_f \ ,$$

where ε_μ is the γ polarization vector and $\bar{u}_i$ and u_f are the bispinors of the initial and final state nucleon.

If particle polarizations are not known the amplitude squared is summed over polarizations of final particles and averaged over polarization of initial particles.

$$|A|^2 = \frac{1}{4}\sum_{\alpha j m} \varepsilon^{*\alpha}_{\mu}\varepsilon^{\alpha}_{\nu}\bar{u}^{(j)}_i A^*_{\mu} u^{(m)}_f \bar{u}^{(m)}_f A_{\nu} u^{(j)}_i \ ,$$

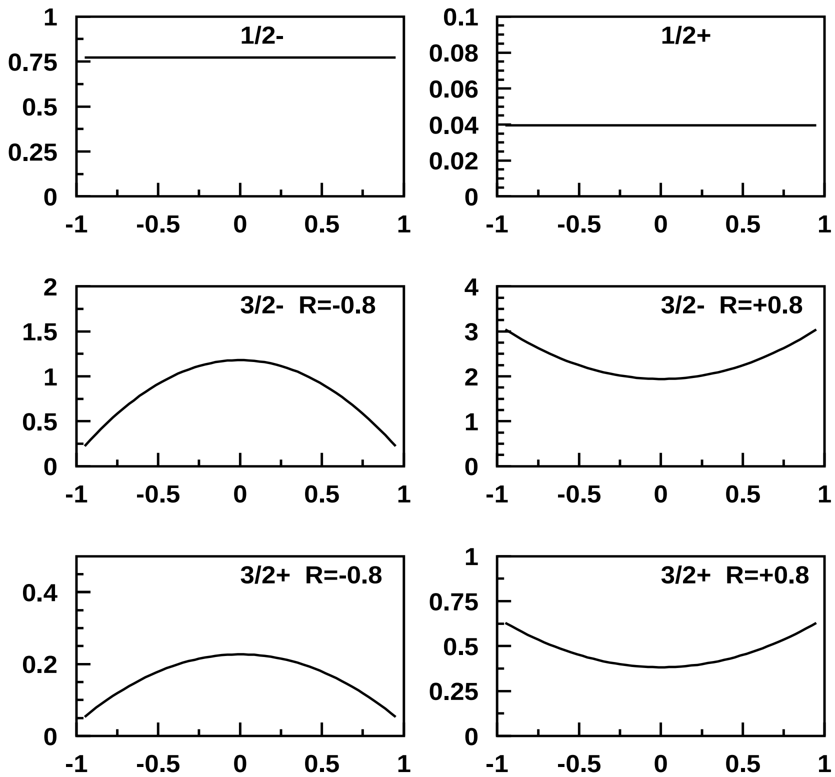
$$\sum_{i=1}^{2} u^{(i)}(k_1)\bar{u}^{(i)}(k_1) = m + \hat{k}_1 \rightarrow (m+\hat{k}_1)\left(1 - i\gamma_5\hat{S}\right)$$

$$\varepsilon^1_{\mu} = (0;1,0,0) \qquad\qquad \varepsilon^2_{\mu} = (0;0,1,0)$$

For non-polarized case:

$$\frac{1}{2}\sum_{\alpha}\varepsilon^{*\alpha}_{\nu}\varepsilon^{\alpha}_{\mu} \quad = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{matrix} Polarization \\ along\, y\, axis \\ \longrightarrow \end{matrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

Single meson photoproduction



1) Meson-meson scattering:

only one observable is measured

2) The πN elastic scattering:

3 observables should be measured for a complete experiment.

3) Meson photoproduction experiment:

8 observables should be measured for a complete experiment.

NN - scattering

Transition of two baryons with momenta p_1 and p_2 into two baryons with p'_1 and p'_2, $s=(p_1+p_2)^2=(p'_1+p'_2)^2$, $k=p_1-p_2$, $k'=p'_1-p'_2$:

$$A=\Big(\bar u(p'_1)V^{S',L'}_{\mu_1\ldots\mu_J}(k'_\perp)u^c(-p'_2)\Big)\,O^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_n}\,\big(\bar u^c(-p_2)V^{S,L}_{\nu_1\ldots\nu_J}(k_\perp)u(p_1)\big)\,A_{pw}(s)\,.$$

$$u^c_j(-p)\;=\;C\bar u^T_j(p)\qquad C=\gamma_2\gamma_0=\begin{pmatrix}0 & -\sigma_2\\ -\sigma_2 & 0\end{pmatrix}$$

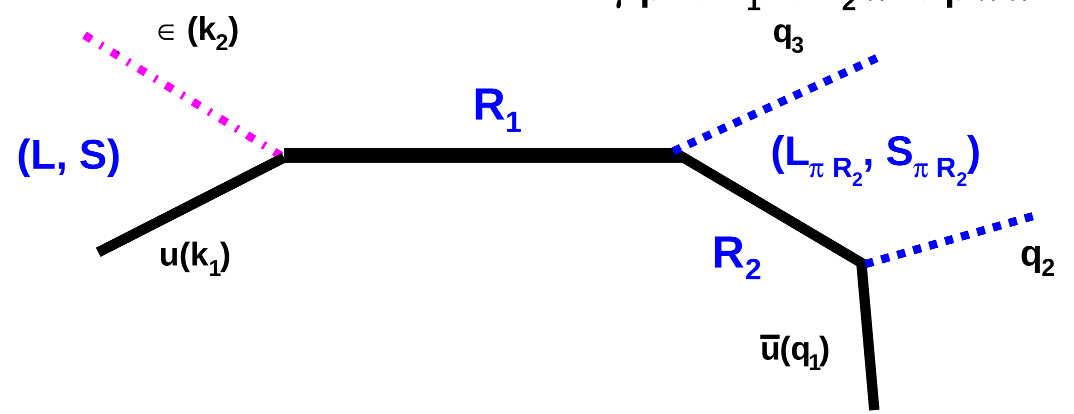
Vertex operators:

$$V^{0,L}_{\mu_1\ldots\mu_J}=i\gamma_5X^{(J)}_{\mu_1\ldots\mu_J}(k^\perp)\qquad V^{1,L<J}_{\mu_1\ldots\mu_J}=\gamma_{\mu_1}X^{(n-1)}_{\mu_2\ldots\mu_J}(k^\perp)$$

$$V^{1,L=J}_{\mu_1\ldots\mu_J}=\varepsilon_{\mu_1\eta\xi\gamma}\gamma_\eta X^{(J)}_{\xi\mu_2\ldots\nu_J}(k^\perp)P_\gamma\qquad V^{1,L>J}_{\mu_1\ldots\mu_J}=\gamma_\alpha X_{\alpha\mu_1\ldots\mu_J}(k^\perp)$$

The resonance amplitudes for meson photoproduction

$\gamma\, p \to R_1 \to R_2\, \pi \to p\, \pi\, \pi$



The general form of the angular dependent part of the amplitude:

$$\bar{u}(q_1)\tilde{N}_{\alpha_1\ldots\alpha_n}(R_2\to\mu N)F^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_n}(q_1+q_2)\tilde{N}^{(j)\beta_1\ldots\beta_n}_{\gamma_1\ldots\gamma_m}(R_1\to\mu R_2)$$
$$F^{\gamma_1\ldots\gamma_m}_{\xi_1\ldots\xi_m}(P)V^{(i)\mu}_{\xi_1\ldots\xi_m}(R_1\to\gamma N)u(k_1)\varepsilon_\mu$$

$$F^{\mu_1\ldots\mu_L}_{\nu_1\ldots\nu_L}(p) = (m+\hat{p})O^{\mu_1\ldots\mu_L}_{\alpha_1\ldots\alpha_L}\frac{L+1}{2L+1}\;\; g^{\perp}_{\alpha_1\beta_1} - \frac{L}{L+1}\sigma_{\alpha_1\beta_1}\;\; \prod_{i=2}^{L} g_{\alpha_i\beta_i}O^{\beta_1\ldots\beta_L}_{\nu_1\ldots\nu_L}$$

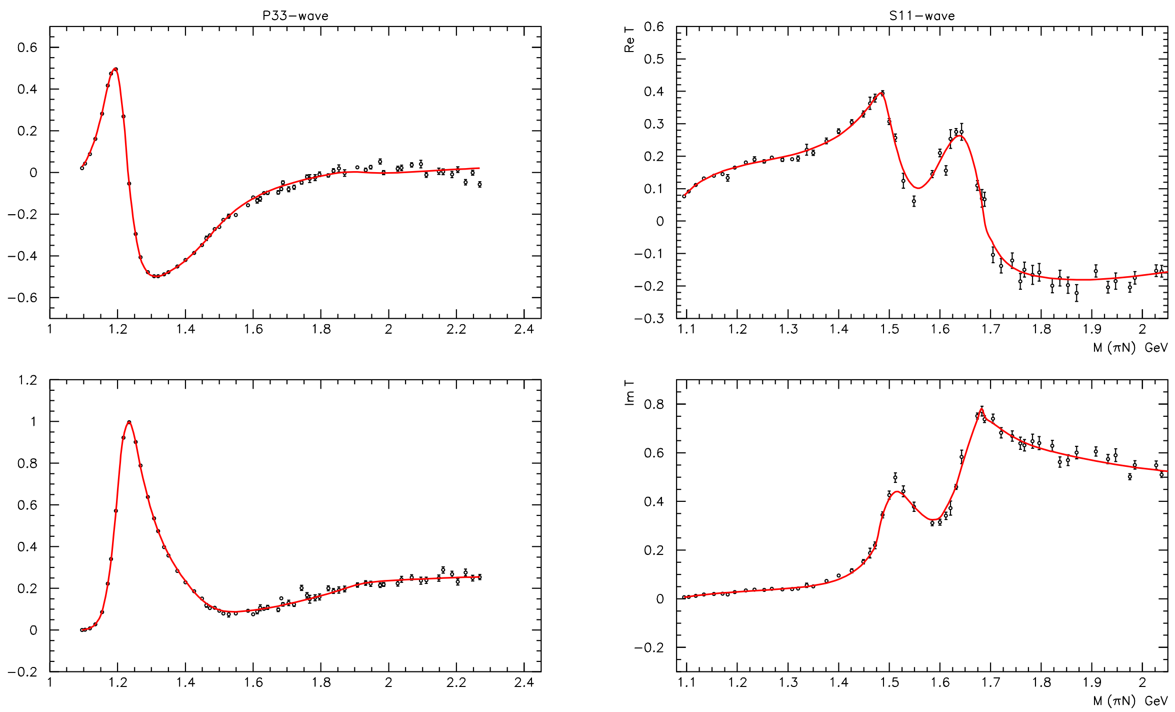
$$\sigma_{\alpha_i\alpha_j} = \frac{1}{2}(\gamma_{\alpha_i}\gamma_{\alpha_j} - \gamma_{\alpha_j}\gamma_{\alpha_i})$$

Energy independent analysis (the πN elastic scattering)

Energy part of partial amplitudes are fitted as parameters

$$A_{\pi N} = \omega^* \left[G(s,t) + H(s,t) i(\vec{\sigma}\vec{n}) \right] \omega'$$

$$G(s,t) = \sum_L \left[(L+1) F_L^+(s) - L F_L^-(s) \right] P_L(z) \qquad H(s,t) = \sum_L \left[F_L^+(s) + F_L^-(s) \right] P_L'(z)$$

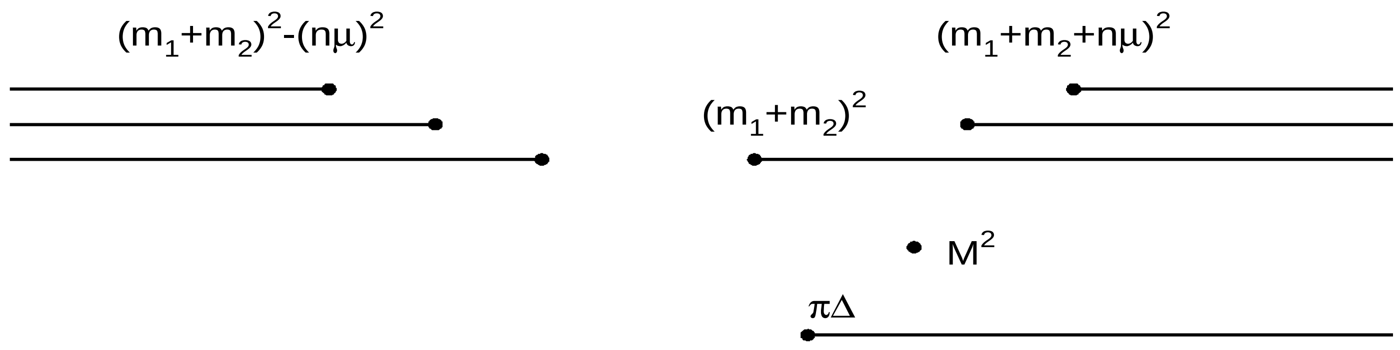


The strong signals can be extracted from existing data imposing dispersion relations.

Energy dependent analysis

Extraction of leading singularities

1. **Pole singularities: stable particles and resonances.**
2. **Threshold (square root) singularities defined by the decay of the system into final particles.**
3. **Logarithmical singularities due to rescattering of three particles (triangle diagrams).**
4. **Box singularities (one over square root) defined by 4 particle rescattering (box diagrams).**
5. **cuts on left-hand side complex plane due to exchange processes.**



The simplest parameterization of the pole, Breit-Wigner amplitude:

$$A = \frac{\Lambda}{M^2 - s - iM\Gamma}$$

The pole is at $s = M^2 - iM\Gamma$. The residue in the pole $R = \Lambda$, the amplitude has a peak at $s = M^2$.

The width of the state is formed by decays into open channels. Then the threshold singularities should be taken into account:

$$A_{ab} = \frac{g_a g_b}{M^2 - s - i\sum\limits_j \rho_j(s) g_j^2}$$

where $\rho_j(s)$ is the phase volume.

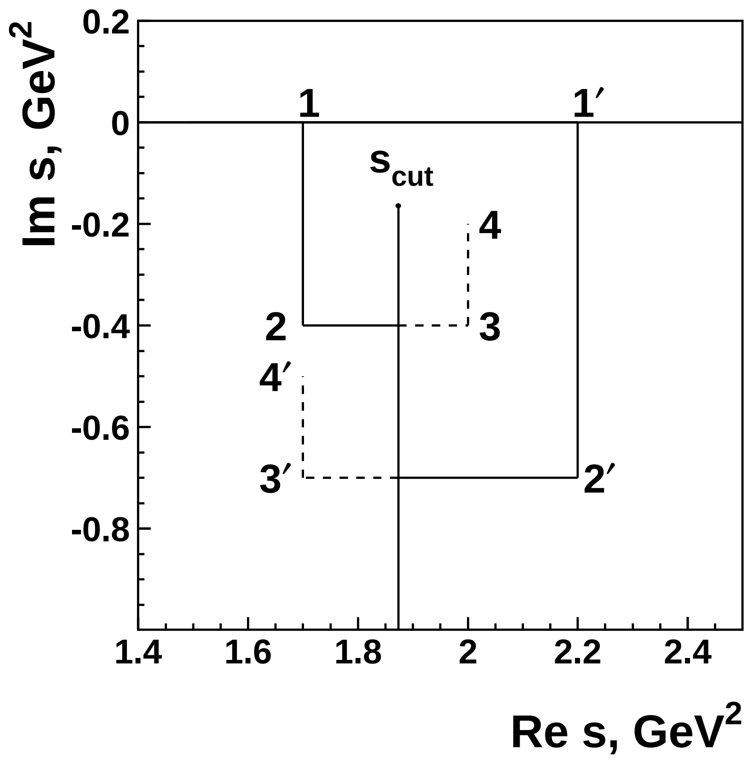
Two body phase volume:

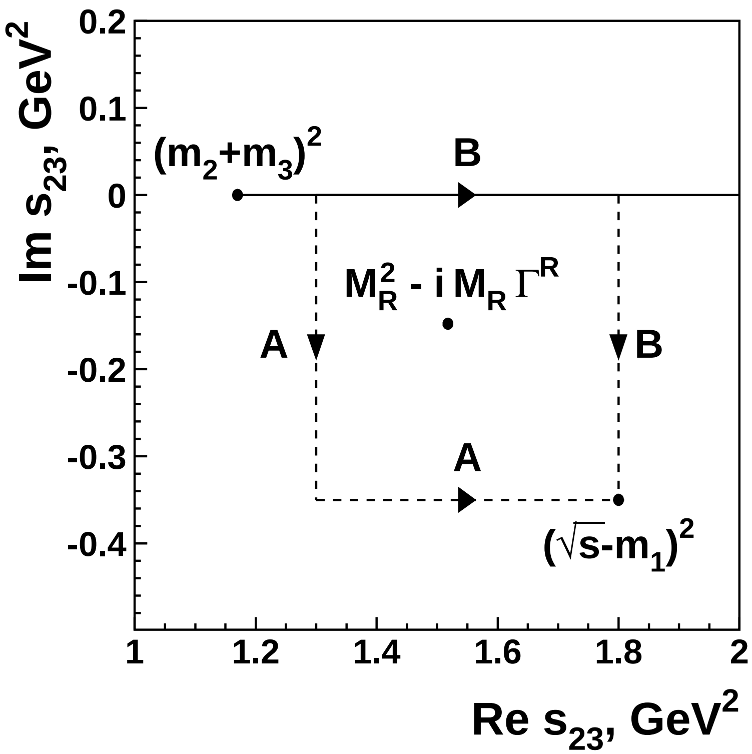
$$\rho(s, m_1, m_2) = \frac{\sqrt{(s-(m_1+m_2)^2)(s-(m_1-m_2)^2)}}{s} \frac{k^{2L}}{F(L, k^2, r)}$$

Three body phase volume:

$$\rho_3(s) = \int\limits_{(m_2+m_3)^2}^{(\sqrt{s}-m_1)^2} \frac{ds_{23}}{\pi} \frac{\rho(s,\sqrt{s_{23}},m_1)\, M_R\Gamma^R_{tot}}{(M_R^2-s_{23})^2+(M_R\Gamma^R_{tot})^2}\ ,$$

$$M_R\Gamma^R_{tot} = \rho(s_{23},m_2,m_3)g^2(s_{23})\ ,$$





3 K-matrix representation of the scattering amplitude

The unitarity condition for the partial wave amplitude:

$$SS^{+} = I \qquad\qquad S = I + 2i\hat{\rho}(s)\hat{A}(s)$$

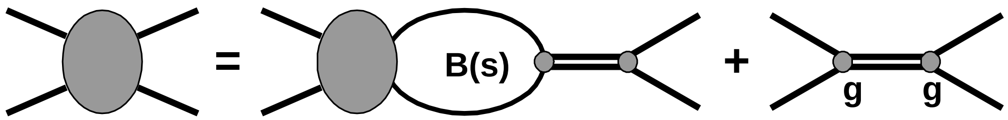
$$S = \frac{I + i\hat{\rho}\hat{K}}{I - i\hat{\rho}\hat{K}} = I + 2i\hat{\rho}A(s), \qquad\qquad A(s) = \hat{K}(I - i\hat{\rho}\hat{K})^{-1}$$

Where $\hat{K}$ is a real matrix.

One pole, multi-channel K-matrix corresponds to the relativistic Breit-Wigner amplitude:

$$K_{ab} = \frac{g_a g_b}{M^2 - s} \qquad \rightarrow \qquad A_{ab} = \frac{g_a g_b}{M^2 - s - i\sum\limits_j \rho_j(s) g_j^2}$$

The K-matrix amplitude can be considered as a solution of Bethe-Salpeter equation:



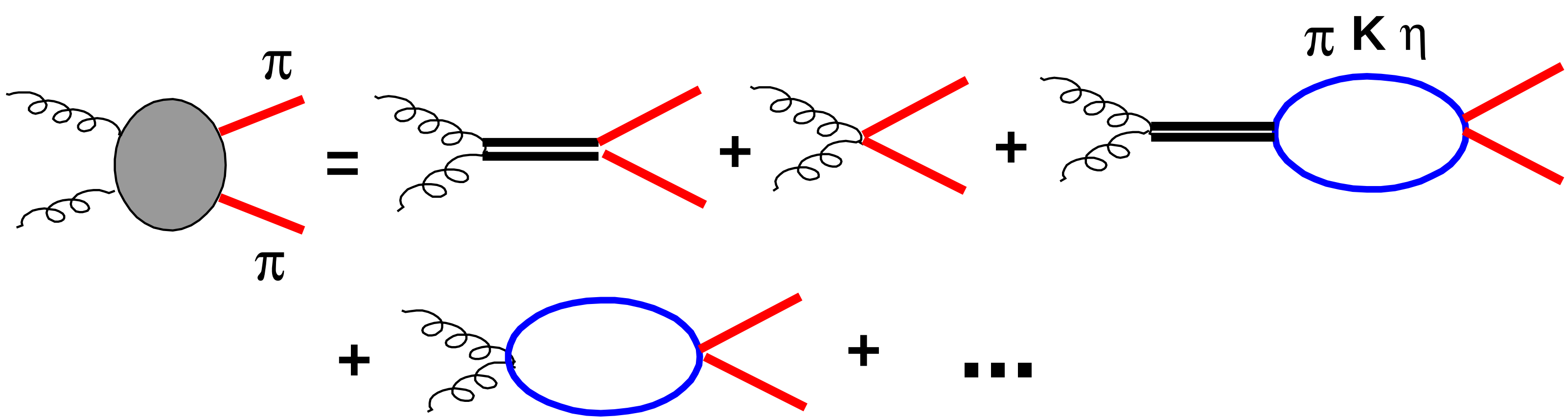
$$A_{ab}(s,s) = \int\limits_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{A_{aj}(s,s') i\rho_j(s') K_{jb}(s)}{s'-s-i0} + K_{ab}(s)$$

But ... with omitted real part of loop diagrams:

$$A_{ab} = A_{aj} i\rho_j(s) K_{jb} + K_{ab} \qquad \rightarrow \qquad \hat{\mathbf{A}} = \hat{\mathbf{K}}(\mathbf{I} - \mathbf{i}\hat{\rho}\hat{\mathbf{K}})^{-1}$$

P-vector approach

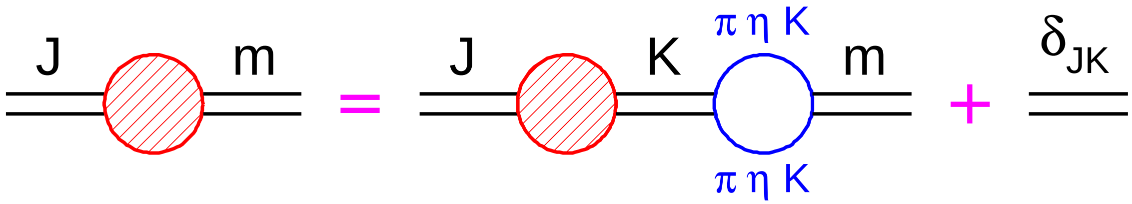
The $\gamma\gamma \to \pi\pi$ reaction: the contribution from the $\gamma\gamma$-loop to the width of the state can be neglected.



$$A_k = P_j (I - i\rho K)^{-1}_{jk} \qquad P_j = \sum_m \frac{\Lambda_n g_1^{(n)}}{M_n^2 - s} + F_j$$

N/D based analysis of the data

In the case of resonance contributions only we have factorization and Bethe-Salpeter equation can be easily solved:



$$A_{jm}=A_{jk}\sum_{\alpha}B^{km}_{\alpha}(s)\frac{1}{M_m-s}+\frac{\delta_{jm}}{M_j^2-s}\qquad B^{km}_{\alpha}(s)=\int\limits_{4m_j^2}^{\infty}\frac{ds'}{\pi}\frac{g^{(k)}_{\alpha}(s')\rho(s')g^{(m)}_{\alpha}(s')}{s'-s-i0}$$

$$\hat{A}=\hat{\kappa}(I-\hat{B}\hat{\kappa})^{-1}\qquad \kappa_{ij}=\frac{\delta_{ij}}{M_i^2-s}\qquad B^{ij}=\sum_{\alpha}B^{km}_{\alpha}(s)$$

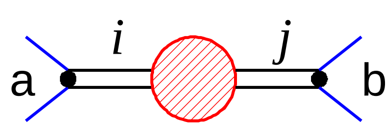
For non-resonant contributions: there is no factorization and the amplitude can have a complicated energy dependence. However in majority of K-matrix analysis the non-resonant contributions are constant or have a simple energy dependence. Non-factorization can be taken into account by introduction of two transitions with fixed left and right vertices.

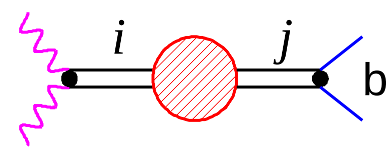
Parameterization of P_{13} wave: 3 resonances 8 channels, 4 non-resonant contributions $\pi N \to \pi N$, $\pi N \to \eta N$, $\pi N \to K\Sigma$, $\pi N \to \Delta\pi$. It corresponds to 8×8 channel K-matrix and 5×5 N/D-matrix.

In many cases (fixed form-factor or subtraction procedure) the real part can be calculated in advance (for S-wave):

$$B(s) = ReB(M^2) + \frac{g^2}{\pi}[\rho(s)\; ln\frac{1-\rho(s)}{1+\rho(s)} - \rho(M^2)\; ln\frac{1-\rho(M^2)}{1+\rho(M^2)}] + i\rho(s)g^2$$

The P-vector approach is strait forward:

$A_{ab} = \sum\limits_{ij}$ 

$P_b = \sum\limits_{ij}$ 

1. **This approach satisfies analyticity and two body unitarity conditions. It takes left-hand side singularities into account.**
2. **The approach is suitable for the analysis of high statistic data in combined analysis of many reactions.**
3. **However: a treatment of the real part for interfering resonances is model dependent.**

Methods for parameter optimization

1. **The two body final states** $\pi N \to \pi N$, $\pi\pi \to \pi\pi$, $\gamma p \to \pi N$, $p\bar{p}(at\,rest) \to 3\pi$**:** χ^2 **method. For** n **measured bins we minimize**

$$\chi^2 = \sum_j^n \frac{(\sigma_j(PWA) - \sigma_j(exp))^2}{(\Delta\sigma_j(exp))^2}$$

2. **Reactions with three or more final states are analyzed with logarithm likelihood method. The minimization function:**

$$f = - \sum_j^{N(data)} ln \frac{\sigma_j(PWA)}{\sum\limits_m^{N(rec\,MC)} \sigma_m(PWA)}$$

This method allows us to take into account all correlations in many dimensional phase space.